3 The Real Business Cycle Model

1 What are Business Cycles?

1.1 A bit of History

The study of business cycles has a pretty long History. Back in the nineteenth century, several engineers and GPs(!) studied the movements of output and prices over some periods of time. Joseph Schumpeter proposed to define cycles as the succession of 4 major phases

1. Expansion;
2. Crisis;
3. Recession;
4. Recovery.

It was then common to use the following classification of cycles

- Juglar (Clément) cycles: their average length is of about 7 to 11 years, and is mainly related to the cycle of fixed investment. This was back then "the" standard way of thinking about the Business cycle.
- Kuznets (Simon) cycles: These cycles have an average length of 15 to 25 years and are associated to the movements in investment in infrastructures (buildings, ...).
- Kondratiev (Nikolaï) cycles: these cycles are way much longer and rather refer to periods of about 45 to 60 years. These were usually thought of as movements in the technology.

These cycles really corresponded to an engineer's view of the economy, and were actually associated as recurrent movements in the main economic variables. These led to trigonometric decompositions of the business cycle, which actually correspond to the development of thermodynamics and Fourier decomposition technics. These technics decompose a time series in a sum of trigonometric functions that all capture particular frequencies — exactly like the physicist decomposes sounds using harmonics. In other words, the behavior of a time series was thought of as the result of an equation of the type

$$y_t = \sum(\alpha_i \cos(it) + \beta_i \sin(it))$$

where each i corresponds to a particular frequency of the business cycle. This view led to the development of models in which the business cycle is the result of totally predictable movements, like Samuelson's oscillator model. This view is however problematic as it leads to a purely deterministic view of the business cycle which is not supported by the empirical evidence.[1]

This led the two economists Arthur F. Burns and Wesley C. Mitchell to propose an alternative definition of Business Cycles in 1946:

> "*Business cycles are a type of fluctuation found in the aggregate economic activity of nations that organize their work mainly in business enterprises: a cycle consists of expansions occurring at about the same time in many economic activities, followed by similarly general recessions, contractions, and revivals which merge into the expansion phase of the next cycle; in duration, business cycles vary from more than one year to ten or twelve years; they are not divisible into shorter cycles of similar characteristics with amplitudes approximating their own.*"

This definition looks pretty similar to the earlier one. It however led to a totally different view of the business cycles. They are not thought of anymore as recurrent movements in activity but as *fluctuations* and acquire a *stochastic* dimension they did not have previously. We are now talking about *amplitudes* and *co–movements* which can be actually measured relying on statistical technics. What remains to be established is how we should measure the business cycle.

1.2 Measuring Business Cycles

The basic idea underlying the measurement of business cycles is that any time series $\{x_t\}_{t=0}^{T}$ can be decomposed into *(i)* a trend component x_t^{T} and *(ii)* a cyclical component x_t^{C}, such that

$$x_t = x_t^{\mathrm{T}} + x_t^{\mathrm{C}}$$

One problem immediately emerges from this representation: identification! How can we identify 2 component given that we only observe one time series? The solution to this problem is actually rather simple. We have to impose some restrictions on what we call the trend component and what we call the cyclical component. The bad news is then that there exists an infinite number of ways to identify a cyclical and a trend component. Just to give you an idea of how easy it is to come up with several decompositions, let us just review some of them.

Output Gap: This definition of the business cycle views the cyclical component as the gap existing between *actual output* and *natural output*. Should actual output be above (below) than natural output, the economy is said to be in expansion (recession). This vision of the cycle

[1]It is however possible to reconcile this view by introducing some stochastic shocks in the basic model.

is the standard Keynesian view of business cycles. Although this seems quite appealing, it is far from being satisfactory for one very basic reason. Natural output is very difficult *1)* to define and *2)* to measure, this second problem being fundamentally related to the first one. What is *natural output*?

- A first approach — the traditional one — is to define it as *potential output*. Potential output corresponds to the level of output that could be achieved in an economy that works at full capacity utilization. This is however something that nobody will ever observe. It is however possible to estimate a production function including utilization rates and then build a counterfactual assuming that these utilization rates are equal to unity. But this approach is highly sensitive to model mis–specification, since it relies on the specification of the technology.
- A second approach defines *natural output* as the level of output that would prevail in an economy with flexible prices and full competition. This approach suffers the same problems as the first one.

It should however be noted that this approach was massively used in most macro models in the 70s and that natural level of output were estimated relying on "technological frontier" estimation technics. It is however too model specific to be applied in a general case.

Unobserved Component Models: This approach relies on filtering technics developed for the estimation of state–space models. The idea is to impose some restrictions directly on the process of the trend and the cyclical component. For instance, starting from the model

$$x_t = x_t^{\mathrm{T}} + x_t^{\mathrm{C}}$$

one imposes that the trend component is a random walk with drift

$$x_t^{\mathrm{T}} = \mu + x_{t-1}^{\mathrm{T}} + \varepsilon_t^{\mathrm{T}}$$

while the cyclical component is an ARMA process of the form

$$\Phi(L)x_t^{\mathrm{C}} = \Theta(L)\varepsilon_t^{\mathrm{C}}$$

where $\varepsilon_t^{\mathrm{T}}$ and $\varepsilon_t^{\mathrm{C}}$ are two uncorrelated gaussian white noises. These process can then be simply estimated relying on maximum likelihood technics. Although attractive, this approach raises an important issue: the cyclical component is again subject to mis–specification errors and the correct specification of the model ought to change over time in case of a structural break in the data, which would lead to a change in the definition of the cyclical component itself.

Related to this approach is the Beveridge–Nelson's [1989] approach which proposes to estimate the ARIMA representation of a time series as

$$\Phi(L)\Delta x_t = \Theta(L)\varepsilon_t$$

where $\Delta x_t = X - t - x_{t-1} = (1-L)x_t$. Beveridge and Nelson then propose to obtain the trend vs cyclical decomposition of the time series relying on the long–run properties of the rational polynomial $H(L) = \Theta(L)/\Phi(L)$.

$$\Delta x_t = \frac{\Theta(L)}{\Phi(L)}\varepsilon_t = H(L)\varepsilon_t = H(1)\varepsilon_t + \frac{H(L)-H(1)}{(1-L)}\Delta\varepsilon_t$$

which can be rewritten as

$$\Delta x_t = \Delta x_t^{\mathrm{T}} + \Delta x_t^{\mathrm{C}}$$

where

$$\Delta x_t^{\mathrm{T}} = H(1)\varepsilon_t \text{ and } \Delta x_t^{\mathrm{C}} = \frac{H(L)-H(1)}{(1-L)}\Delta\varepsilon_t$$

such that

$$x_t^{\mathrm{T}} = x_{t-1}^{\mathrm{T}} + H(1)\varepsilon_t \text{ and } x_t^{\mathrm{C}} = \frac{H(L)-H(1)}{(1-L)}\varepsilon_t$$

Like the UC approach, this one is highly sensitive to mis–specification errors. However the Box and Jenkins methodology is helpful to get a pretty good representation of the data. The Beveridge and Nelson's actually identifies the trend component as that component that is generated by long–run fluctuations only, the rest being allocated to the cyclical component. This has two main drawbacks. First of all, this leads to a very highly volatile trend component, which does not correspond to our understanding of the long–run. Second, it allocates two many components to the cyclical component — for instance medium run phenomena will be allocated to the cyclical component.[2]

Deterministic trend: This approach is rather straightforward as it rests on the identification of the trend component as a time polynomial. In other words, the trend component is obtained by fitting the following equation

$$x_t = \sum_{i=0}^{p} \alpha_i t^i + u_t$$

the trend component, x_t^{T} is then obtained as

$$x_t^{\mathrm{T}} = \sum_{i=0}^{p} \widehat{\alpha}_i t^i$$

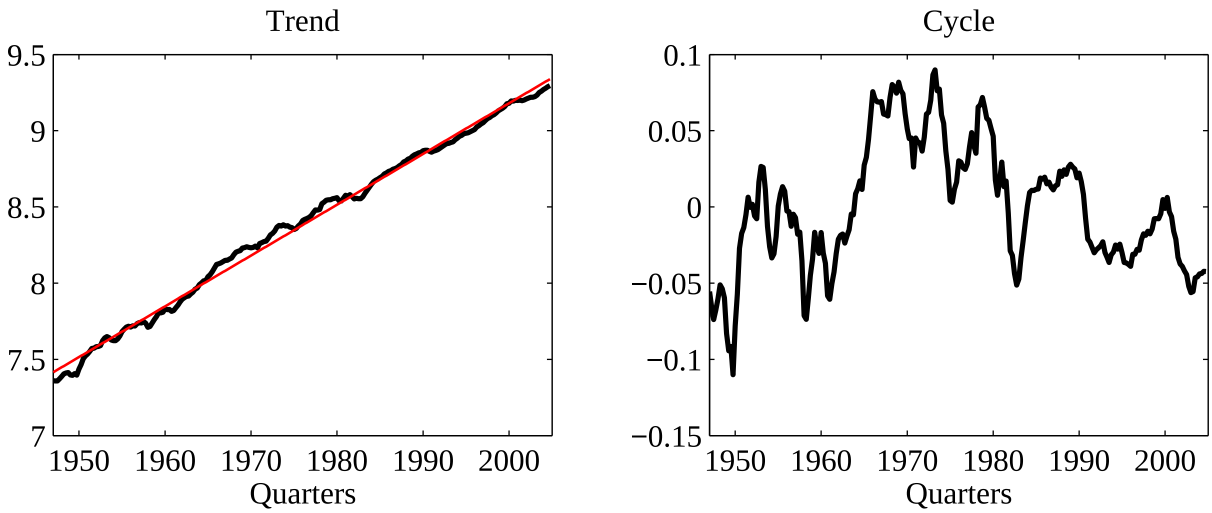
[2] Note that if $H(1) = 1$ this corresponds to a cycle defines in terms of growth rates if x_t represents the logarithm of an aggregate.

and the cyclical component is given by

$$x_t^{\mathrm{C}} = u_t$$

The common practice is to use a linear trend. The main reason for this choice is that this corresponds to the basic prediction of a standard exogenous growth model. Figure 1, illustrates this approach on US output data. An expansion is then define as a situation where actual output

Figure 1: US cycle: Linear Trend



is above the deterministic trend. One potential drawback with this approach is that the trend component, as defined by the linear trend may not be flexible enough to capture all changes in the trend. But more importantly, it assumes that the trend component is fundamentally deterministic. An direct implication of this result is that any stochastic aspect of the time series is attributed to the trend component. This may however lead to strong biases in the evaluation of the cyclical component. Indeed, should the economic trend be stochastic, as it would be the case if the trend were determined by a random walk with drift, this aspect of the trend component would be mis–allocated to the cyclical component.

It is therefore very easy to come up with various definitions of the trend and cyclical components. It is also clear from the previous discussion that no definition is any better than any other one. In any case, what seems to be needed is a way to obtain a method that

- is flexible enough to capture changes in the trend component;
- allows for the trend to be stochastic;
- eliminates most of the long–run phenomena from the cyclical component.

A method was proposed by Hodrick and Prescott [1980].[3]

The Hodrick–Prescott Filter: In 1980, Hodrick and Prescott (HP) proposed to use a filter identifies the cyclical component as the gap between the actual series and the trend component. Their approach is essentially non–parametric and imposes a minimum of restrictions on what is called the trend. The idea of the method is simple: it starts by identifying the trend component and builds the cyclical component as the difference between the actual series and the identified trend. The trend is identified by imposing two restrictions:

1. The trend should track the actual series.
2. The trend should be smooth.

The first part of the identification is obtained by minimizing the distance between the actual series, x_t, and its trend component, x_t^{T}. The second part of the identification is obtained by constraining changes in the slope of the trend. Therefore, the trend component is obtained by solving the program

$$\min_{\{x_\tau^{\mathrm{T}}\}_{\tau=1}^t} \sum_{\tau=1}^{t} (x_\tau - x_\tau^{\mathrm{T}})^2$$

subject to

$$\sum_{\tau=2}^{t-1} \left((x_{\tau+1}^T - x_\tau^T) - (x_\tau^T - x_{\tau-1}^T) \right)^2 \leqslant c$$

which can be rewritten as a Lagrangian problem as

$$\min_{\{x_\tau^{\mathrm{T}}\}_{\tau=1}^t} \sum_{\tau=1}^{t} (x_\tau - x_\tau^{\mathrm{T}})^2 + \lambda \sum_{\tau=2}^{t-1} \left((x_{\tau+1}^{\mathrm{T}} - x_\tau^{\mathrm{T}}) - (x_\tau^{\mathrm{T}} - x_{\tau-1}^{\mathrm{T}}) \right)^2$$

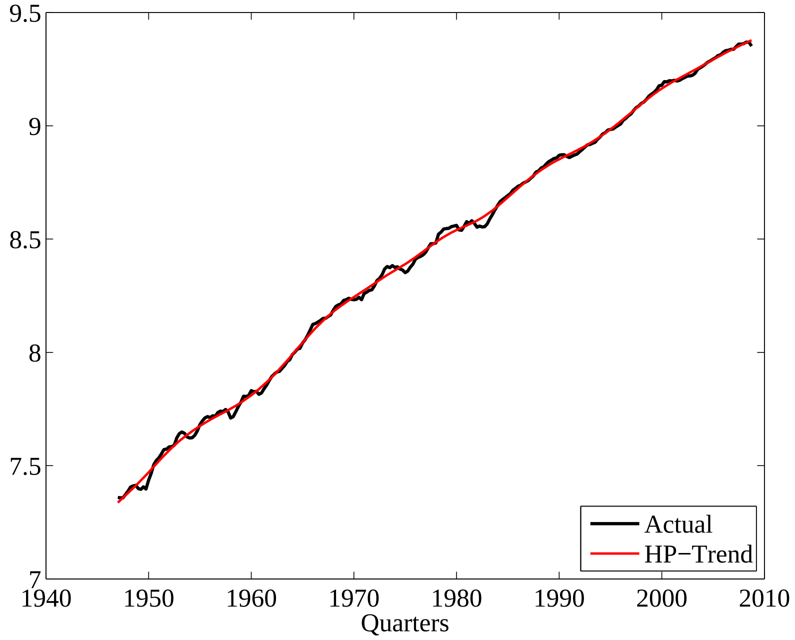
where λ is the Lagrange multiplier associated to the constraint. Note that when λ is set to zero, the trend is identical to the actual series. When λ is set to ∞, any change in the slope is infinitely costly. Therefore the slope is constant, and we get back a linear trend. The question is then how to set λ? A consensus has emerged regarding a reasonable value for λ.[4] As soon as we are to use quarterly data, λ is set such that we accept cyclical variations up to 5% per quarter, and changes in the quarterly rate of growth of 1/8% per quarter. This lead to a value of λ of 1600. Figure 2 reports the HP trend and cyclical component for quarterly US real GDP.

[3] "A" method because, one may obviously come up with an infinite number of filters: bandpass filters, moving averages, ...

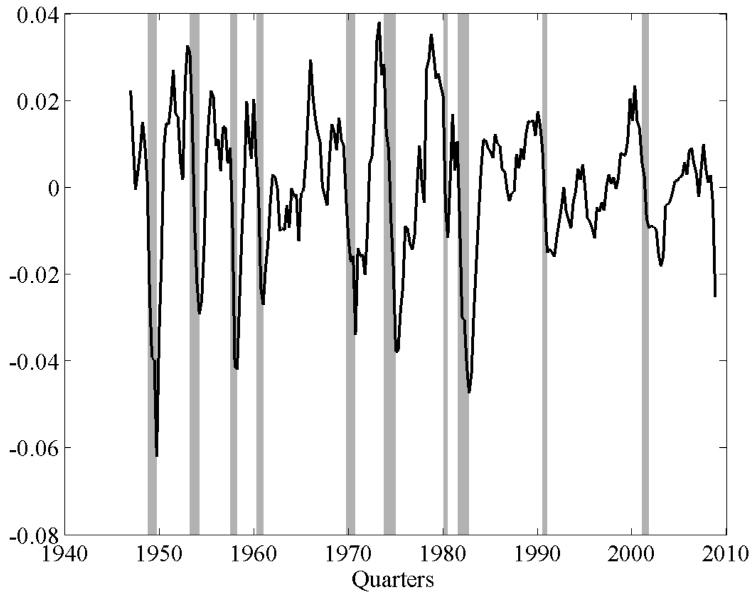
[4] Note however that this consensus is totally *ad hoc*.

Figure 2: HP filtered US output

(a) Actual Output vs HP–Trend Component



(c) HP–Cyclical Component



The sample runs from 1947:I to 2004:IV. Panel (a) displays the actual series (dark plain line) and the HP trend component (red line). It clearly shows that the trend is not exactly linear, but appears to be smooth. Panel (b) of the figure reports the cyclical component of output. Shaded areas corresponds to US recessions as identified by the National Bureau of Economic research (NBER). As can be seen the so identified cyclical component tracks remarkably well the recessions identified by the NBER.

1.3 Characterizing the Cycle

Once we have identify the cyclical component it is possible to characterize the business cycle. What is at stake is the behavior of "*recurrent fluctuations of macroeconomic aggregates about trend*". In other words, we would like to be able to obtain some regularities in the data — some stylized facts — that will help us to understand the business cycles. This amounts to get information on the joint distribution of the fluctuations of the cyclical component of output, consumption, investment ... This can be simply measured by computing some statistical moments

- *volatilities*, to measure the amplitude of the business cycle
- *correlations*, to measure the co–movements of the main macroeconomic aggregates

As a first attempt to visualize the type of results we should get, let us have a look at the evolution of time series. Figures 3–5 report the evolution of the cyclical component of the main aggregates over the business cycle. Before having a look at the graphs, it is however important to precisely define the variables we will have a look at. First of all, we will focus on the US economy (although in tutorials, we will replicate the analysis for the Australian economy) for the period 1947:I–2004:IV. All data are obtained from the *Federal Reserve Economic Data* and are downloadable from `http://research.stlouisfed.org/fred2/` and are real data. Variables are defined as follows

- Consumption (C): Nondurables + Services;
- Investment (I): Durables + Fixed Investment + Changes in inventories;
- Government Consumption (G): Total Government Consumption;
- Output: Real Gross Domestic Product;
- Labor: Hours Worked;
- Productivity: Output/Hours Worked.

Figure 3: Cyclical Component of Consumption

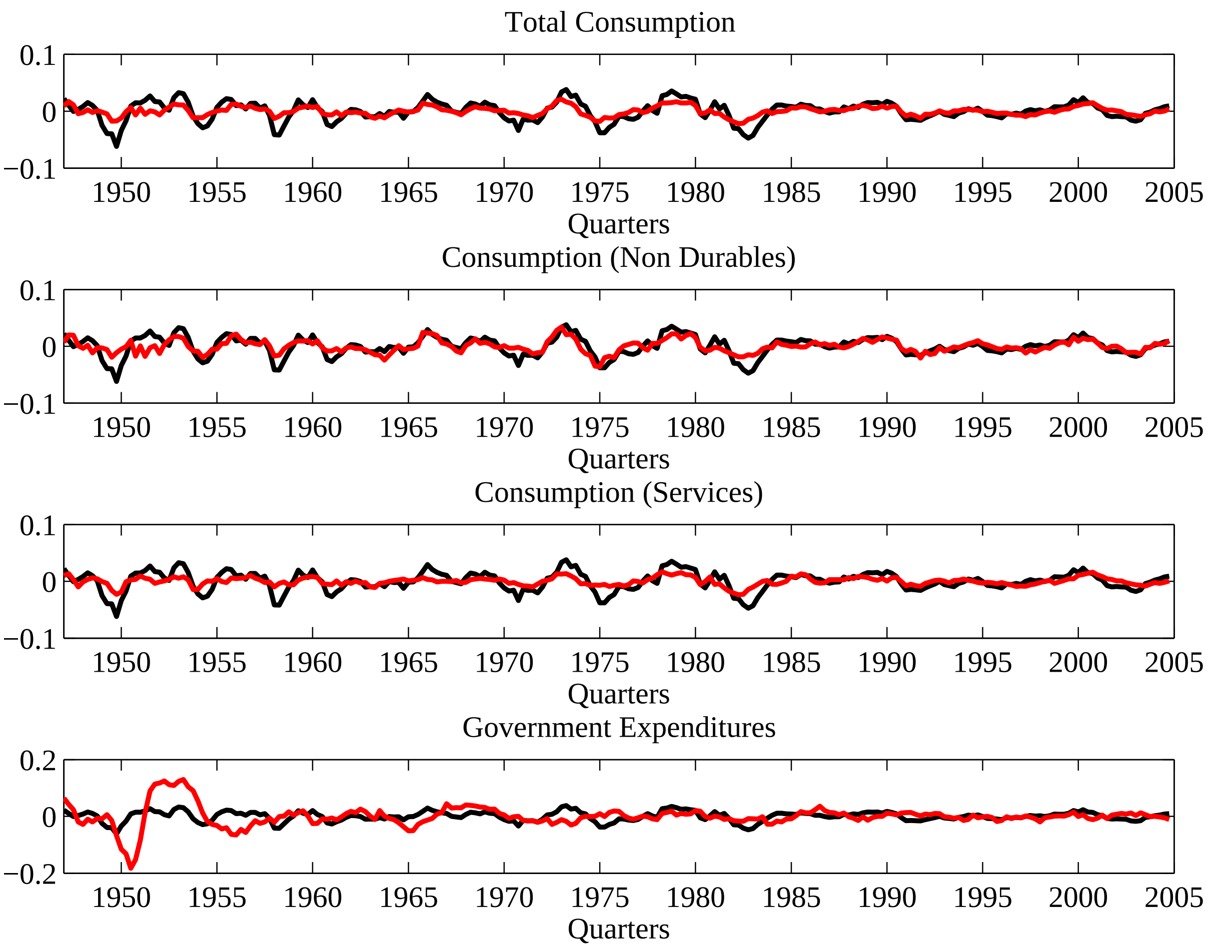


Figure 4: Cyclical Component of Investment

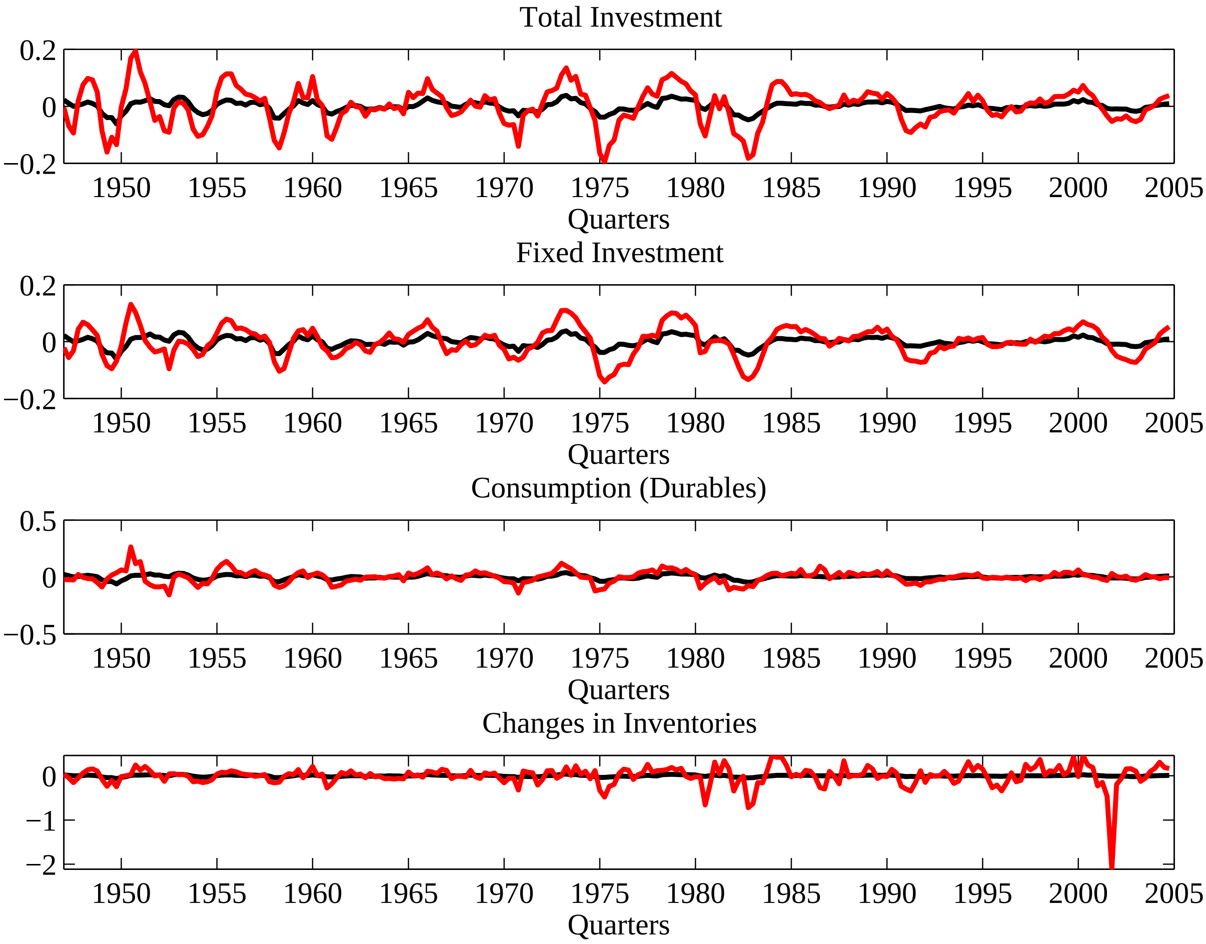
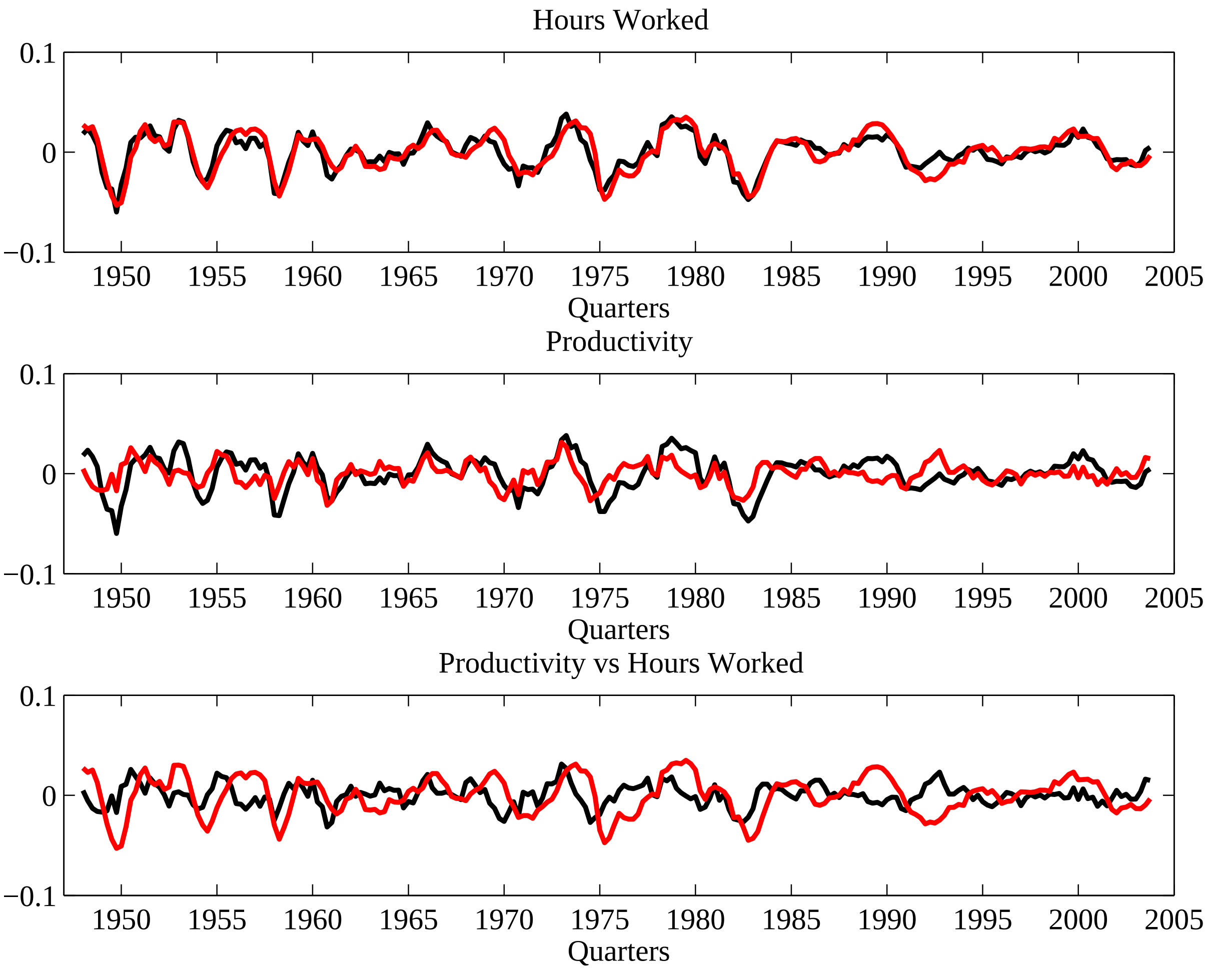


Figure 5: Cyclical Component of the Labor Market



From the observation of the figures, it can be seen that

- Consumption of non-durables is less volatile than output.
- Consumption of services is less volatile than output;
- And therefore so is total consumption.
- All three variables are positively correlated with output;
- Consumption of durables is more volatile than output;
- Private investment is much more volatile than output.
- Changes in inventories are much more volatile than output.
- And therefore so is total investment.
- All four variables are positively correlated with output;
- Government expenditures is volatile with large changes in times of war. In times of peace, government expenditures are less volatile than output.
- Government expenditures are very weakly correlated with output;
- Hours worked are as or more volatile than output, and are positively correlated with output. In fact, employment is way more volatile on the extensive margin than on the intensive margin.
- Labor Productivity is less volatile than output, and seems positively correlated with output

All these observations translate into moments, as reported in Table 1. Consumption is found to be half volatile as output, while investment is four times more volatile. Hours worked are as volatile as output, while the volatility of labor productivity volatility is half of that of output. All aggregates are found to be positively correlated with output, however it is worth noting that productivity is much less procyclical. All variables exhibit persistence. Finally labor productivity and hours worked are not correlated. These observations are called *stylized facts*, because they are common to most industrialized countries and are robust to the way we measure aggregates. For instance, should one just focus on non–durables or services, consumption is always less volatile than output. This finding is robust to the period, the country and the frequency. To borrow from Lucas, "*Business Cycles are all alike!*"

These are the regularities that the Real Business Cycle model attempts to account for.

Table 1: HP–Filtered Moments

Variable	$\sigma(\cdot)$	$\sigma(\cdot)/\sigma(y)$	$\rho(\cdot, y)$	$\rho(\cdot, h)$	Auto(1)
Output	1.70	–	–	–	0.84
Consumption	0.80	0.47	0.78	–	0.83
Services	1.11	0.66	0.72	–	0.80
Non Durables	0.72	0.42	0.71	–	0.77
Investment	6.49	3.83	0.84	–	0.81
Fixed investment	5.08	3.00	0.80	–	0.88
Durables	5.23	3.09	0.58	–	0.72
Changes in inventories	22.48	13.26	0.48	–	0.40
Hours worked	1.69	1.00	0.86	–	0.89
Labor productivity	0.90	0.53	0.41	0.09	0.69

2 The Real Business Cycle Model

The Real Business Cycle research program follows up on a suggestion made by Lucas [1980] who argued that economists

> "[...] need to test them (models) as useful imitations of reality by subjecting them to shocks for which we are fairly certain how actual economies or parts of economies would react. The more dimensions on which the model mimics the answers actual economies give to simple questions, the more we trust its answers to harder questions."

The RBC model which we now present is a perfect example of this approach to economics. We first give an attempt as to how we can come up with the idea of building such a model. We then present the theoretical model.

2.1 From the Data to the Model: A Crash Course in Economic Modeling

How to build a model?

This is basically the question we will now address in this section. We will try to use the data to identify the basic mechanisms that we would like to see in the model and therefore obtain a skeleton of a first potential candidate.

Let us start from the first three observations we made: consumption is less volatile than output, investment is more volatile than output and both consumption and investment are procyclical. What can be learned from these three observations? Let us assume that we model a close economy. In such a context, we know that in equilibrium, investment is financed by savings. We therefore face a situation in which consumption is less volatile than output, while savings are more volatile than output. Meanwhile savings and consumption are both positively with output — *i.e.* with income. We immediately see that consumption/savings dynamics are dictated by smoothing motives. In other words, we would like to have a model that builds on the permanent income model.

A second lesson we can take from these observations is that we need a model in which there is capital accumulation, since we have investment! In other words, we will have a model in which asset holdings ought to take the form of capital ownership. Capital shall then be useful — *i.e.* capital will be an input in a production function. In other words, we will have to model technology.

The labor market brings us also a lot of information regarding the model we should build. Hours worked are as volatile as output and productivity is less volatile than output. Hours are strongly procyclical, while productivity is weakly procyclical and almost orthogonal to hours worked. If we are to think at a simple micro model of labor supply, we know that leisure is a *normal good*. An implication of this result is that consumption and leisure are positively correlated, while the data suggest the opposite: since hours worked are procyclical, leisure should be countercyclical. This is a very important piece of information as it is telling us that the demand side should be important to shape good properties for hours worked. Another important information that the data deliver is that productivity and labor are almost orthogonal. However, if we consider the basic production function, any increase in labor yields a decrease in productivity (by diminishing returns). This therefore suggests that as labor increases, another phenomenon drives labor productivity upward. We will see that the introduction of technology shocks will generate this effect.

From these very basic and casual observations, we can now infer that we will need a permanent income model for the households' behavior and that we will have firms in the model. Both firms and households should take endogenous labor decisions. It is then clear that the model we will develop will be a general equilibrium model.

2.2 The Model

The baseline RBC model builds on the optimal growth model proposed by Cass [1965] and extended to a stochastic environment by Brock and Mirman [1972]. However, rather than to an optimal growth model, one should think about the baseline RBC model as the neo–classical model of capital accumulation. This section will present the baseline RBC model as proposed by King, Plosser and Rebelo [1988]. This should be thought of as a minimal version of the model which does not feature any of the developments that we will introduce next.

The economy is comprised of a continuum of identical households of unit mass. The households consume, supply labor and accumulate physical capital that they rent to firms. There is a continuum of identical and perfectly competitive firms that rent capital and labor services from the households at market determined prices. Firms produce an homogenous good by means of capital and labor according to a constant return technology. In each and every period, the economy is hit ny technology shocks that shift the production function. Hence, agents have to take their decisions under uncertainty. These shocks will be the main driver of economic fluctuations, as agents will optimally respond to these shocks. Despite the economy will be Pareto optimal, we will present the decentralized version of the model, in order to make clear the decisions of the agents.

The Households: The economy is populated by a continuum of identical households who are all infinitely lived. Two comments are in order

1. First of all, it may sound rather weird to consider that agents are infinitely–lived. However, we may simply think about of a dynasty of agents who just all care about their descendants. Hence, when making their choice, these agents actually take into account how their decisions will affect the utility of their children, who also care about their own children ... One way to model that is to assume that the utility that an agent achieves in a given period t, $V(c_t)$, takes the following form

$$V_t = u(c_t) + \beta \mathbb{E}_t V_{t+1}$$

in other words, the agent's utility in period t depends on the utility he gets when consuming c_t plus what he expects his children to enjoy in the next period. This rewrites

$$V_t = u(c_t) + \beta \mathbb{E}_t [u(c_{t+1}) + \beta \mathbb{E}_{t+1} V_{t+2}]$$

which rewrites

$$\begin{aligned} V_t &= u(c_t) + \beta \mathbb{E}_t [u(c_{t+1})] + \beta^2 \mathbb{E}_t [V_{t+2}] \\ &= u(c_t) + \beta \mathbb{E}_t [u(c_{t+1})] + \beta^2 \mathbb{E}_t [u(c_{t+2})] + \beta^3 \mathbb{E}_t [V_{t+3}] \\ &\vdots \\ &= \mathbb{E}_t \left[\sum_{\tau=0}^{\infty} \beta^\tau u(c_{t+\tau}) \right] \end{aligned}$$

We then end up with an infinitely lived agent problem.

2. Second of all, assuming that all agents are identical is a rather strong assumption. However, we have to keep in mind the all purpose of this model: explain aggregate fluctuations. We are not interested in distributional issues. Such that taking into account heterogeneity would unnecessarily complicate the model. One may however argue that the presence of heterogeneity can affect the properties of the economy at the aggregate level. Krusell and Smith [1998] showed that this is not the case. Therefore, since all agents are identical and heterogeneity does not matter, we will adopt the fiction of a *representative agent*.

The representative household has preferences over a consumption bundle, C_t, and leisure, ℓ_t. Instantaneous preferences are represented by the utility function:

$$U(C_t, \ell_t) \tag{1}$$

Note that the household values consumption **and** leisure time, which makes the model depart from the baseline permanent income model. This is important as this will lead to an endogenous labor supply decision. It turns out that the labor supply will be a key element of the model. The utility function $U(.,.)$ is assumed to satisfy

(H1) *The utility function* $U : \mathbb{R}_+ \times [0,1] \longrightarrow \mathbb{R}$ *is of class* $\mathcal{C}^2$*, strictly increasing and concave. It satisfies the following Inada conditions.*

$$\lim_{C\to 0} \frac{\partial U(C,\ell)}{\partial C} = \infty, \ \lim_{\ell\to 0} \frac{\partial U(C,\ell)}{\partial \ell} = \infty,$$
$$\lim_{C\to \infty} \frac{\partial U(C,\ell)}{\partial C} = 0, \ \lim_{\ell\to \infty} \frac{\partial U(C,\ell)}{\partial \ell} = 0.$$

Furthermore, King et al. [1988] show that the only mathematical form that will be compatible with the existence of a balanced growth path is given by

$$U(C_t,\ell_t) = \begin{cases} \left(\dfrac{C_t^{1-\sigma}-1}{1-\sigma}\right) v(\ell_t) & \text{if } \sigma \in \mathbb{R}^+\backslash\{1\} \\ \log(C_t) + v(\ell_t) & \text{if } \sigma = 1 \end{cases}$$

Since our representative agent is infinitely lived, he values consumption and leisure over his entire life cycle, such that his expected discounted utility is given by

$$\mathcal{U}_t = E_t\left[\sum_{\tau=0}^{\infty} \beta^\tau U(C_{t+\tau}, \ell_{t+\tau})\right] \tag{2}$$

where $\beta \in (0,1)$ denotes the psychological discount factor of the agent —*i.e.* the way this agent values the future. The exact form of the intertemporal utility function calls for two main comments

1. Utility is valued using information available in period t. This information will consists of the aggregate stock of physical capital available in the economy, the current shocks, and the distribution of the shocks.

2. Preferences are assumed to be time–separable. Note that this assumption can be relaxed very easily. For example one may assume some form of habit persistence either in consumption or leisure, or may assume some temporal dependency in the discount factor.

In each and every period, the household faces two constraints. The first one restricts his time allocation decision

$$\ell_t + h_t \leqslant 1 \tag{3}$$

This constraint states that the household shall allocate his total time endowment —that we normalize to unity without loss of generality— between leisure time, ℓ_t, and productive activities, h_t.

The second constraint is the budget constraint he faces in each and every period

$$\underbrace{B_t}_{\text{Bond purchases}} + \underbrace{C_t + I_t}_{\text{Good purchases}} \leqslant \underbrace{(1+r_{t-1})B_{t-1}}_{\text{Bond revenus}} + \underbrace{W_t h_t}_{\text{Wages}} + \underbrace{z_t K_t}_{\text{Capital revenus}}$$

The agent enters period t holding B_{t-1} bonds purchased from other agents at the end of period $t-1$ and the capital stock K_t. He gets the return, r_t, on the bonds he holds and rents the flow of capital services to the firms at price z_t. Finally, he gets the wage, W_t, from the labor he supplies on the labor market. These revenues are used to purchase goods for consumption, C_t, and investment, I_t, purposes[5] and new bonds from the other agents. The agent invests in order to accumulate physical capital. The law of motion of capital is given by

$$K_{t+1} = I_t + (1-\delta)K_t \tag{4}$$

where $\delta \in [0,1]$ is the depreciation rate of the capital stock. This calls for several comments

1. By allowing the household to accumulate capital and rent it to firms, we considerable simplify the model as it is not necessary to specify formally the asset markets. In fact, Modigliani–Miller's theorem holds in this economy, such that the mode of financing of capital does not matter. The firm just rents the capital stock from the household, which amounts to make the firm purchase one particular good: the flow of capital service. The rental price then depends on the state of the business cycle, and everything is observationally equivalent to a situation where agents hold contingent claims that insure themselves against any risk and whose support is the value of the firm —*i.e.* the capital stock. In this sense, markets are complete.

[5] Note that the good serves as a numéraire in this economy.

2. The depreciation rate is assumed to remain constant over the business cycle, which is a rather strong assumption. We will evaluate the implications of relaxing this assumption.

Note that since the utility is strictly increasing in both arguments, the budget constraint and the time allocation constraint will hold with equality. Plugging the latter and the law of motion of capital in the budget constraint, we get

$$B_t + K_{t+1} + C_t = (1 + r_{t-1})B_{t-1} + (z_t + 1 - \delta)K_t + W_t h_t$$

Let us denote $\Omega_t = B_{t-1} + K_t$ the wealth of the agent, then the budget constraint rewrites

$$\Omega_{t+1} + C_t = (z_t + 1 - \delta)\Omega_t + [(1 + r_{t-1}) - (z_t + 1 - \delta)]B_{t-1} + W_t h_t$$

The problem of the representative household is therefore given by

$$\max_{\{C_{t+\tau},\Omega_{t+\tau+1},h_{t+\tau},B_{t+\tau}\}} \mathbb{E}_t \left[\sum_{\tau=0}^{\infty} \beta^\tau U(C_{t+\tau}, (1 - h_{t+\tau})) \right]$$

subject to

$$\Omega_{t+1} + C_t = (z_t + 1 - \delta)\Omega_t + [(1 + r_{t-1}) - (z_t + 1 - \delta)]B_{t-1} + W_t h_t$$

The first order conditions associated to this program are given by

$$\begin{aligned}
C_t &: \frac{\partial U(C_t, \ell_t)}{\partial C_t} = \Lambda_t \\
h_t &: \frac{\partial U(C_t, \ell_t)}{\partial \ell_t} = \Lambda_t W_t \\
\Omega_{t+1} &: \Lambda_t = \beta \mathbb{E}_t \left[\Lambda_{t+1}(z_{t+1} + 1 - \delta) \right] \\
B_t &: (1 + r_t)\mathbb{E}_t[\Lambda_{t+1}] = \mathbb{E}_t \left[\Lambda_{t+1}(z_{t+1} + 1 - \delta) \right]
\end{aligned}$$

Eliminating Λ_t, the system reduces to

$$\frac{\partial U(C_t, \ell_t)}{\partial \ell_t} = \frac{\partial U(C_t, \ell_t)}{\partial C_t} W_t \tag{5}$$

$$\frac{\partial U(C_t, \ell_t)}{\partial C_t} = \beta \mathbb{E}_t \left[\frac{\partial U(C_{t+1}, \ell_{t+1})}{\partial C_{t+1}} (z_{t+1} + 1 - \delta) \right] \tag{6}$$

$$(1 + r_t)\mathbb{E}_t \left[\frac{\partial U(C_{t+1}, \ell_{t+1})}{\partial C_{t+1}} \right] = \mathbb{E}_t \left[\frac{\partial U(C_{t+1}, \ell_{t+1})}{\partial C_{t+1}} (z_{t+1} + 1 - \delta) \right] \tag{7}$$

to which we should add the transversality condition

$$\lim_{s \to +\infty} \mathbb{E}_t \left[\beta^s \frac{\partial U(C_{t+s}, \ell_{t+s})}{\partial C_{t+s}} \Omega_{t+s+1} \right] = 0 \tag{8}$$

Equation (5) determines the labor supply behavior of the household. By allocating a marginal unit of hours in productive activities, the household reduces its leisure time and therefore suffers a utility loss of $\frac{\partial U(C_t,\ell_t)}{\partial \ell_t}$. However, by supplying its labor in the productive sector he earns extra wage, W_t, which enables him to buy additional consumption good which increases its utility by $\frac{\partial U(C_t,\ell_t)}{\partial C_t}W_t$. The household supplies labor until the gains and costs are equalized.

Equation (6) teaches us the consumption/saving behavior of the household. Assume the household is given an extra unit of good today. He can obviously consume it immediately and therefore enjoy an extra utility of $\frac{\partial U(C_t,\ell_t)}{\partial C_t}$. But he can also decides to postpone consumption and invest this extra unit of good to form capital. He will then rent this capital to the firm and therefore earn a return $z_{t+1}+1-\delta$ per unit of capital, which will enable him to purchase additional consumption good and therefore enjoy a utility gain of $\frac{\partial U(C_{t+1},\ell_{t+1})}{\partial C_{t+1}}(z_{t+1}+1-\delta)$. This is obtained for a given value of the shocks tomorrow, and obviously shocks are drawn from a distribution, such that the gains the household can expect is given by $\mathbb{E}_t\left[\frac{\partial U(C_{t+1},\ell_{t+1})}{\partial C_{t+1}}(z_{t+1}+1-\delta)\right]$. Since this gain will be earned tomorrow, it has to be expressed in period t units, which is achieved by applying the discount factor. In other words, should the left hand side of the equation be greater (lower) than the right hand side, the household will consume (invest) the extra unit of good. Optimal behavior is achieved when no arbitrage is left. Note that this behavior exactly corresponds to the consumption smoothing behavior.

Equation (7) tells us that the return on assets should be equal to the return on capital. Should the return on assets be larger, the household would not invest and would only hold bond. The opposite would hold if the inequality is reversed. In other words, this is a non–arbitrage condition that determines the allocation of the household portfolio.

The last condition, Equation 8, states that in the infinite of time, the household will not accumulate wealth. One way to actually understand this condition is to think about an agent with finite life. Assume our agent is totally selfish and aware that he will die in the next period. In this case, his best interest is to totally eat his remaining wealth, such that in the next period nothing remains. Hence, the utility gain of accumulation should be nil. Since our agent never dies, this will only occur in the limit. Viewed from today, this transversality condition states that the discounted expected utility from wealth accumulation is nil in the infinite of time. This actually gives us a terminal condition for the problem of the agent.

Using equation (5) and (6), we obtain

$$\frac{\partial U(C_t,\ell_t)}{\partial \ell_t}\frac{1}{W_t}=\beta\mathbb{E}_t\left[\frac{\partial U(C_{t+1},\ell_{t+1})}{\partial \ell_{t+1}}\frac{1}{W_{t+1}}(z_{t+1}+1-\delta)\right]$$

To understand this relationship, let us again consider that the household is given one extra unit of the good. Since, the wage income is expressed in terms of good, $1/W_t$ corresponds to the transformation rate that converts units of the good in units of time. In other words, by applying $1/W_t$, the household can instantaneously turn good into leisure time, which yields him extra utility $\frac{\partial U(C_t,\ell_t)}{\partial \ell_t}$. However, he can invest this extra unit of good and build some capital that he will rent to the firm. This will yields further extra discounted income $\beta(z_{t+1}+1-\delta)$, which he will be able to turn into leisure applying next period transformation rate $1/Wt+1$. He will then be able to enjoy more leisure time and get extra utility $\frac{\partial U(C_{t+1},\ell_{t+1})}{\partial \ell_{t+1}} \frac{1}{W_{t+1}}$. Since tomorrow is uncertain, he has to consider his decision based on expectations. In other words, the household smoothes out his leisure consumption and therefore his labor supply.

The Firms: Just like there exists a representative household, we assume there exists a representative firm, which produces the homogenous good by means of capital and labor services according to a constant returns to scale technology. The technology is represented by the production function

$$Y_t = A_t F(K_t, \Gamma_t h_t) \tag{9}$$

where Γ_t denotes Harrod neutral technological progress,[6] which is assumed to grow at the exogenous constant gross rate $\gamma > 1$:

$$\Gamma_t = \gamma \Gamma_{t-1}$$

Note that King et al. [1988] showed that this form of technological progress is the only one which is compatible with a balanced growth path. The production function satisfies

(H2) *The production function $F : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}^+$ is a function of class $\mathcal{C}^2$, strictly increasing, quasi–concave and homogenous of degree 1. It satisfies the following Inada conditions:*

$$\lim_{K\to 0} \frac{\partial F(K,N)}{\partial K} = \infty, \quad \lim_{N\to 0} \frac{\partial F(K,N)}{\partial N} = \infty,$$
$$\lim_{K\to \infty} \frac{\partial F(K,N)}{\partial K} = 0, \quad \lim_{N\to \infty} \frac{\partial F(K,N)}{\partial N} = 0.$$

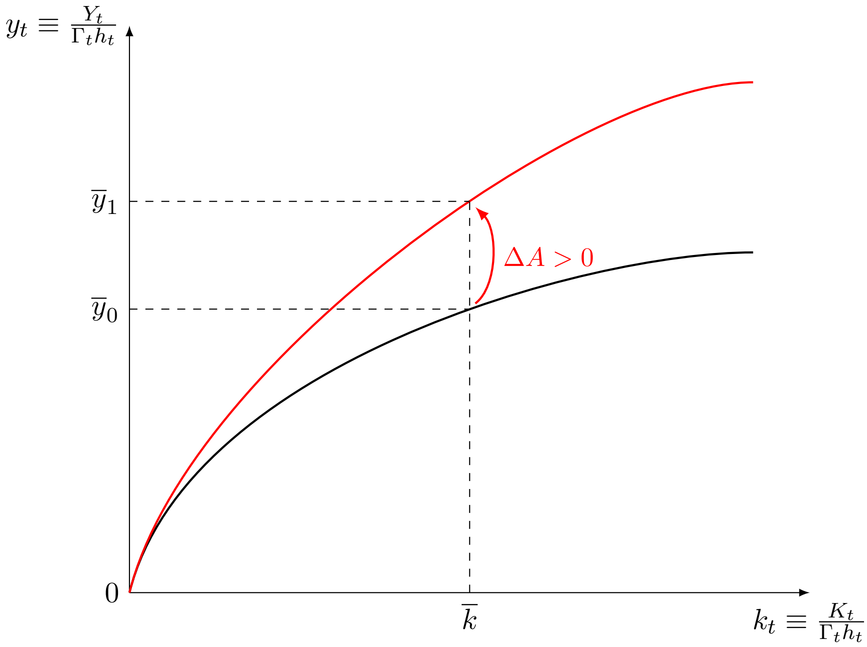
Finally the two inputs are assumed to be essential: $F(K,0) = F(0,N) = 0$.

A_t is a stochastic shock that hits the *total factor productivity* in each and every period. A positive shock on A_t implies that, for a given level of inputs used in the production process, more output will be produced. This is depicted in figure 6. This figure reports the level of output per efficient hour worked $(Y_t/(\Gamma_t h_t)$ as a function of the capital labor ratio $(K_t/(\Gamma_t h_t)$. For a given capital labor ratio, $\overline{k}$, an increase in A_t shifts the level of output upward (from $\overline{y}_0$ to $\overline{y}_1$). Thus the temporary component to productivity is assumed to follow an autoregressive process of order one (*i.e.* an AR(1)) process of the form

$$\log(A_t) = \rho \log(A_{t-1}) + (1-\rho)\log(\overline{A}) + \varepsilon_t \tag{10}$$

[6] Technological progress is said to be Harrod neutral if for a given rental price of capital, technological progress leaves the capital labor ratio unchanged. In other words, Harrod neutral technological progress is the one that affects the labor input.

Figure 6: Effect of a Technology shock



where $|\rho| < 1$ and ε is a gaussian iid shock with mean 0 and standard deviation σ_a.

The firm then determines its production plan by maximizing its profit

$$\Pi(K_t, h_t) = A_t F(K_t, \Gamma_t h_t) - W_t h_t - z_t K_t$$

which leads to the first order conditions

$$A_t \frac{\partial F(K_t, \Gamma_t h_t)}{\partial K_t} = z_t \tag{11}$$

$$A_t \frac{\partial F(K_t, \Gamma_t h_t)}{\partial h_t} = W_t \tag{12}$$

General equilibrium: We are now in a position to define the general equilibrium of this economy.

Definition 1 *A general equilibrium of this economy is a sequence of prices* $\mathscr{P} = \{z_t, W_t, r_t\}_{t=0}^{\infty}$, *and a sequence of quantities,* $\mathscr{Q} = \{C_t, I_t, Y_t, K_{t+1}, h_t, \ell_t, B_t\}_{t=0}^{\infty}$ *such that*

1. *Given a sequence of prices* $\mathscr{P}$, *the sequence of quantities* $\mathscr{P}$ *solves the households and the firm problem.*

2. *Given a sequence of quantities* $\mathscr{Q}$, *the sequence of prices* $\mathscr{P}$ *clears all markets.*

Let us first realize that since agents are identical, they have no reason to trade with each other (no trade theorem). In other words, they have no reasons to hold these bonds, B_t, we introduced in the model: $B_t = 0$. This implies that the budget constraint of an agent simplifies to

$$C_t + I_t = W_t h_t + z_t K_t$$

Since we have perfect competition and the technology exhibit constant returns to scale, we have

$$Y_t = W_t h_t + z_t K_t$$

Hence, using the budget constraint of the representative household and the last condition, we get the resource constraint of the economy

$$Y_t = C_t + I_t$$

The conditions characterizing the general equilibrium are therefore given by[7]

$$\begin{aligned}
U_c(C_t, 1 - h_t) &= \Lambda_t \\
U_\ell(C_t, 1 - h_t) &= \Lambda_t W_t \\
Y_t &= A_t F(K_t, \Gamma_t h_t) \\
W_t &= A_t F_h(K_t, \Gamma_t h_t) \\
z_t &= A_t F_K(K_t, \Gamma_t h_t) \\
Y_t &= C_t + I_t \\
K_{t+1} &= I_t + (1 - \delta) K_t \\
\Lambda_t &= \beta \mathbb{E}_t \left[\Lambda_{t+1} \left(z_{t+1} + 1 - \delta\right)\right]
\end{aligned}$$

together with he law of motion of exogenous variables

$$\begin{aligned}
\Gamma_t &= \gamma \Gamma_{t-1} \\
\log(A_t) &= \rho \log(A_{t-1}) + (1 - \rho) \log(\overline{A}) + \varepsilon_t
\end{aligned}$$

and the transversality condition

$$\lim_{s \to +\infty} \mathbb{E}_t \left[\beta^s \frac{\partial U(C_{t+s}, \ell_{t+s})}{\partial C_{t+s}} \Omega_{t+s+1}\right] = 0$$

What we end up with is therefore a non–linear system of stochastic finite difference equations under rational expectations, which we now have to solve.

[7] From now on, we will denote

$$G_{x_i}(x_1, \ldots, x_i, \ldots, x_n) = \frac{\partial G(x_1, \ldots, x_i, \ldots, x_n)}{\partial x_i}.$$

3 Solving the Model

Obtaining a solution to this model is not easy. In fact it does not admit an analytical solution in the general case. We have to to rely on numerical approximations. There is however one particular case, which admits an analytical solution and which will prove useful to understand the difficulty of the exercise.

3.1 An Analytical Example

Let us first leave aside the existence of growth and therefore assume that $\Gamma = \gamma = 1$.Let us assume that

- the utility function of the agent is of the logarithm type:

$$U(C_t, \ell_t) = \log(C_t) + \theta \log(\ell_t)$$

  with $\theta > 0$.

- the production function of the firm is Cobb–Douglas

$$F(K_t, h_t) = K_t^{\alpha} h_t^{1-\alpha}$$

  with $\alpha \in (0, 1)$.

- there is full depreciation of capital in one period ($\delta = 1$)

In this case, the conditions for a general equilibrium are given by

$$\begin{aligned}
\frac{1}{C_t} &= \Lambda_t \\
\frac{\theta}{1 - h_t} &= \Lambda_t W_t \\
Y_t &= A_t K_t^{\alpha} h_t^{1-\alpha} \\
W_t &= (1 - \alpha)\frac{Y_t}{h_t} \\
z_t &= \alpha \frac{Y_t}{K_t} \\
Y_t &= C_t + I_t \\
K_{t+1} &= I_t \\
\Lambda_t &= \beta \mathbb{E}_t \left[\Lambda_{t+1} z_{t+1}\right]
\end{aligned}$$

together with he law of motion of exogenous variables

$$\log(A_t) = \rho \log(A_{t-1}) + (1 - \rho) \log(\overline{A}) + \varepsilon_t$$

and the transversality condition

$$\lim_{s\to+\infty}\mathbb{E}_t\left[\beta^s\frac{K_{t+1+s}}{C_{t+s}}\right]=0$$

The system reduces to

$$\begin{aligned}
\frac{\theta}{1-h_t} &= \frac{(1-\alpha)}{h_t}\frac{Y_t}{C_t}\\
Y_t &= A_tK_t^{\alpha}h_t^{1-\alpha}\\
Y_t &= C_t+K_{t+1}\\
\frac{1}{C_t} &= \beta\mathbb{E}_t\left[\frac{1}{C_{t+1}}\alpha\frac{Y_{t+1}}{K_{t+1}}\right]
\end{aligned}$$

Remember that K_{t+1} is chosen in period t, such that it is known in that period. This implies that the last equation can be rewritten as

$$\frac{K_{t+1}}{C_t}=\beta\mathbb{E}_t\left[\alpha\frac{Y_{t+1}}{C_{t+1}}\right]$$

Making use of the budget constraint, we get

$$\frac{K_{t+1}}{C_t}=\beta\mathbb{E}_t\left[\alpha\frac{C_{t+1}+K_{t+2}}{C_{t+1}}\right]$$

or

$$\frac{K_{t+1}}{C_t}=\alpha\beta\mathbb{E}_t\left[1+\frac{K_{t+2}}{C_{t+1}}\right]$$

Let us denote $X_t=\frac{K_{t+1}}{C_t}$, the last equation rewrites

$$X_t=\alpha\beta\mathbb{E}_t\left[1+X_{t+1}\right]$$

Iterating forward, we have

$$\begin{aligned}
X_t &= \alpha\beta\mathbb{E}_t\left[1+\alpha\beta\mathbb{E}_{t+1}\left[1+X_{t+2}\right]\right]=\alpha\beta(1+\alpha\beta)+(\alpha\beta)^2\mathbb{E}_t[X_{t+2}]\\
&= \alpha\beta(1+\alpha\beta)+(\alpha\beta)^2\mathbb{E}_t[\alpha\beta\mathbb{E}_t\left[1+X_{t+1}\right]=\alpha\beta(1+\alpha\beta+(\alpha\beta)^2)+(\alpha\beta)^3\mathbb{E}_t[X_{t+3}]\\
&\vdots\\
&= \alpha\beta(1+\alpha\beta+(\alpha\beta)^2+\ldots)+\lim_{j\to\infty}(\alpha\beta)^j\mathbb{E}_t\left[X_{t+j}\right]\\
&= \alpha\beta\sum_{s=0}^{\infty}(\alpha\beta)^s+\lim_{j\to\infty}(\alpha\beta)^j\mathbb{E}_t\left[X_{t+j}\right]
\end{aligned}$$

Recall that the transversality condition writes

$$\lim_{s\to+\infty}\mathbb{E}_t\left[\beta^s\frac{K_{t+1+s}}{C_{t+s}}\right]=\lim_{s\to+\infty}\mathbb{E}_t\left[\beta^sX_{t+s}\right]=0$$

since $\alpha\in(0,1)$, we also have

$$\lim_{j\to\infty}(\alpha\beta)^j\mathbb{E}_t\left[X_{t+j}\right]=0$$

such that

$$X_t = \frac{\alpha\beta}{1-\alpha\beta} \Longleftrightarrow K_{t+1} = \frac{\alpha\beta}{1-\alpha\beta}C_t$$

Using this result in the resource constraint of the economy, we obtain

$$K_{t+1} = \alpha\beta Y_t \text{ and } C_t = (1-\alpha\beta)Y_t$$

Therefore, we have $C_t/Y_t = (1-\alpha\beta)$. Plugging this result in the labor supply decision

$$\frac{\theta}{1-h_t} = \frac{(1-\alpha)}{h_t}\frac{Y_t}{C_t} = \frac{(1-\alpha)}{h_t}\frac{1}{1-\alpha\beta}$$

Solving for h_t, we get

$$h_t = \frac{1-\alpha}{1-\alpha+\theta(1-\alpha\beta)} = \overline{h}$$

Therefore,

$$K_{t+1} = \kappa_k A_t K_t^\alpha \text{ and } C_t = \kappa_c A_t K_t^\alpha$$

where

$$\kappa_k \equiv \alpha\beta\overline{h}^{1-\alpha} \text{ and } \kappa_c \equiv (1-\alpha\beta)\overline{h}^{1-\alpha}$$

Knowing this solution it is then very easy to compute moments and impulse responses. But this version model of the model delivers very counterfactual results: hours are constant (perfect smoothing), output, investment and consumption are perfectly correlated... We shall therefore consider a less constrained version of the model.

3.2 The Baseline Version

We will now deal with the RBC model in its canonical version, as proposed by King et al. [1988]. This version actually imposes two of the assumption of the previous example *(i)* a logarithmic utility function and *(ii)* a Cobb–Douglas production function, but relaxes the assumption of perfect depreciation such that $\delta \in (0,1)$ and assumes that the economy experiences growth ($\gamma > 1$). In this case, the model writes

$$\begin{aligned}
\frac{1}{C_t} &= \Lambda_t \\
\frac{\theta}{1-h_t} &= \Lambda_t W_t \\
Y_t &= A_t K_t^\alpha (\Gamma_t h_t)^{1-\alpha} \\
W_t &= (1-\alpha)\frac{Y_t}{h_t} \\
z_t &= \alpha\frac{Y_t}{K_t} \\
Y_t &= C_t + I_t \\
K_{t+1} &= I_t + (1-\delta)K_t \\
\Lambda_t &= \beta\mathbb{E}_t\left[\Lambda_{t+1}(z_{t+1}+1-\delta)\right]
\end{aligned}$$

together with he law of motion of exogenous variables

$$\Gamma_t = \gamma\Gamma_{t-1}$$
$$\log(A_t) = \rho\log(A_{t-1}) + (1-\rho)\log(\overline{A}) + \varepsilon_t$$

and the transversality condition

$$\lim_{s\to+\infty} \mathbb{E}_t\left[\beta^s \frac{K_{t+1+s}}{C_{t+s}}\right] = 0$$

This version of the model does not admit an analytical solution. We will have to rely on a numerical approach. There are many ways to solve the model numerically. We will rely on a log–linear approximation of the model around the steady state, which will give us a system a stochastic linear difference equations that we now know how to solve. The method involves 5 steps

1. Transform the model into a stationary model;
2. Find the deterministic steady state;
3. Take a log–linear approximation of the model around this deterministic steady state;
4. Assign values to parameters;
5. Solve the system a stochastic linear difference equations.

Deflating the Model: Since we need to take a log–linear approximation of the model around its deterministic steady state, we need to consider a version of the model that admits a steady state! however, as long as we have growth in the model, the economy does not converge to a steady state but keeps growing along a balanced growth path. We therefore need to get rid off the growth component. This is achieved by deflating the variables that grow by this growth component. The first question we have to deal with is: *What are these variables that grow?* Since h_t and ℓ_t are constrained to lie between 0 and 1, these two variables clearly do not grow. On the contrary, consumption, output, investment, capital, the wage are all growing. In fact they are all growing at the same pace dictated by Γ_t. We therefore define

$$x_t = X_t/\Gamma_t \text{ for } x \in (c, i, k, y, w)$$

Λ_t decreases at the same rate as Γ_t, such that we define $\lambda_t = \Gamma_t\Lambda_t$. Finally, z_t does not grow as it is given by the ratio of Y_t and K_t. Using these definitions in the system of equations characterizing the equilibrium, we get

$$\begin{aligned}
\frac{1}{c_t\Gamma_t} &= \frac{\lambda_t}{\Gamma_t}\\
\frac{\theta}{1-h_t} &= \frac{\lambda_t}{\Gamma_t}\Gamma_t w_t\\
\Gamma_t y_t &= A_t(\Gamma_t k_t)^{\alpha}(\Gamma_t h_t)^{1-\alpha}\\
\Gamma_t w_t &= (1-\alpha)\frac{\Gamma_t y_t}{h_t}\\
z_t &= \alpha\frac{\Gamma_t y_t}{\Gamma_t k_t}\\
\Gamma_t y_t &= \Gamma_t c_t + \Gamma_t i_t\\
\Gamma_{t+1}k_{t+1} &= \Gamma_t i_t + (1-\delta)\Gamma_t k_t\\
\frac{\lambda_t}{\Gamma_t} &= \beta\mathbb{E}_t\left[\frac{\lambda_{t+1}}{\Gamma_{t+1}}(z_{t+1}+1-\delta)\right]
\end{aligned}$$

Simplifying by Γ_t and making use of the fact that $\Gamma_{t+1} = \gamma\Gamma_t$

$$\begin{aligned}
\frac{1}{c_t} &= \lambda_t\\
\frac{\theta}{1-h_t} &= \lambda_t w_t\\
y_t &= A_t k_t^{\alpha} h_t^{1-\alpha}\\
w_t &= (1-\alpha)\frac{y_t}{h_t}\\
z_t &= \alpha\frac{y_t}{k_t}\\
y_t &= c_t + i_t\\
\gamma k_{t+1} &= i_t + (1-\delta)k_t\\
\lambda_t &= \frac{\beta}{\gamma}\mathbb{E}_t\left[\lambda_{t+1}(z_{t+1}+1-\delta)\right]
\end{aligned}$$

Solving for the Deterministic Steady State: First of all, you should understand that this model actually admits a stochastic steady state. The economy does not converges to a single point in space, but rather to a distribution as the economy is hit by shocks in each and every period. However, given the solution method we will use, the simple knowledge of the deterministic steady state will be sufficient for our purpose. It is given by

$$\frac{1}{c^\star} = \lambda^\star \tag{13}$$

$$\frac{\theta}{1-h^\star} = \lambda^\star w^\star \tag{14}$$

$$y^\star = A^\star k^{\star\alpha} h^{\star 1-\alpha} \tag{15}$$

$$w^\star = (1-\alpha)\frac{y^\star}{h^\star} \tag{16}$$

$$z^\star = \alpha\frac{y^\star}{k^\star} \tag{17}$$

$$y^\star = c^\star + i^\star \tag{18}$$

$$\gamma k^\star = i^\star + (1-\delta)k^\star \tag{19}$$

$$\lambda^\star = \frac{\beta}{\gamma}\left[\lambda^\star(z^\star + 1 - \delta)\right] \tag{20}$$

Note that for our purpose, we do not need to know the value of each variable, but rather the value of some great ratios like consumption/output, investment/output, capital/output... Equation (20) simplifies to

$$z^\star = \frac{\gamma - \beta(1-\delta)}{\beta}$$

Then, using Equation (17), we get

$$\frac{k^\star}{y^\star} = \frac{\alpha\beta}{\gamma - \beta(1-\delta)}$$

and equation (19) gives us the investment/capital ratio

$$\frac{i^\star}{k^\star} = \gamma + \delta - 1$$

such that

$$\frac{i^\star}{y^\star} = \frac{i^\star}{k^\star} \times \frac{k^\star}{y^\star} = \frac{\alpha\beta(\gamma+\delta-1)}{\gamma - \beta(1-\delta)}$$

Then, from equation (18), we get

$$\frac{c^\star}{y^\star} = 1 - \frac{i^\star}{y^\star}$$

Equations (13), (14) and (16) yield

$$\frac{h^\star}{1-h^\star} = \frac{1-\alpha}{\theta c^\star/y^\star}$$

such that

$$h^\star = \frac{1-\alpha}{1-\alpha+\theta c^\star/y^\star}$$

Once we know $h^\star$, we would be able to recover all quantities in the model. For instance:

$$y^\star = \overline{A}\left(\frac{k^\star}{y^\star}\right)^{\frac{\alpha}{1-\alpha}} h^\star$$

such that it is then easy to get $i^\star$, $c^\star$, $k^\star$...

Taking a Log–Linear Approximation of the Model: It is important that you understand that each equation belonging to the system that characterizes the equilibrium can be written as:

$$\mathbb{E}_t[\varphi(X)] = 0$$

where X denotes the vector of variables dated either t or $t+1$.Let us denote $X^\star$ the value of X in the deterministic steady state. Note that by definition of the steady state, the $\varphi(.)$ function satisfies:

$$\varphi(X^\star) = 0$$

Let us denote $x = \log(X)$, such that each equation actually rewrites

$$\mathbb{E}_t[\varphi(\exp(x))] = 0$$

Taking a log approximation of this equation amounts to take a first order Taylor expansion of the preceding equation with respect to x (rather than X). The first order Taylor expansion of the last equation around the deterministic steady state $x^\star$ is given by:

$$\mathbb{E}_t[\varphi(\exp(x))] = \mathbb{E}_t \left[\varphi(\exp(x^\star)) + \sum_i \left(\frac{\partial \varphi(\exp(x))}{\partial \exp(x_i)} \exp(x_i) \right) \Bigg|_{x=x^\star} (x_i - x_i^\star) + \mathcal{O}(\|x\|^2) \right]$$

where $\mathcal{O}(\|x\|^2)$ is infinitely small in probability. Let us denote $\widehat{x}_i = x_i - x_i^\star$. Note that

$$x_i - x_i^\star = \log(X_i) - \log(X_i^\star) \simeq \frac{X_i - X_i^\star}{X_i^\star}$$

such that $\widehat{x}_i$ can be interpreted as the percentage deviation of x_i from its deterministic steady state value, and the coefficient in front of each of them can be given an interpretation in terms of elasticity. Using the fact that $X_i^\star = \exp(x_i^\star)$, the log linear approximation rewrites

$$\mathbb{E}_t[\varphi(X)] = \mathbb{E}_t \left[\varphi(X^\star) + \sum_i \left(\frac{\partial \varphi(X)}{\partial X_i} X_i \right) \Bigg|_{x=x^\star} \widehat{x}_i + \mathcal{O}(\|x\|^2) \right]$$

Since by definition of the deterministic steady state, we have $\varphi(X^\star) = 0$, this reduces to

$$\mathbb{E}_t[\varphi(X)] = \mathbb{E}_t \left[\sum_i \left(\frac{\partial \varphi(X)}{\partial X_i} X_i \right) \Bigg|_{x=x^\star} \widehat{x}_i + \mathcal{O}(\|x\|^2) \right]$$

Finally, neglecting higher order terms, we end up with

$$\mathbb{E}_t \varphi(X) \simeq \mathbb{E}_t \left[\sum_i \left(\frac{\partial \varphi(X)}{\partial X_i} X_i \right) \Bigg|_{x=x^\star} \widehat{x}_i \right]$$

we then obtain a system of linear stochastic difference equations that we can solve.

? Is this a legitimate procedure?

It is important to understand what we are doing when we take a log–linear approximation to a model. Indeed, getting rid off higher order terms is not innocuous. This amounts to assume that the approximation satisfies a *certainty equivalence property*: volatility does not affect the behavior of the agents. In other words, this assumes that risk does not exert any effect of the behavior of the agents. In particular this means that agents do not form any precautionary savings in this model! *Are we making a big mistake?* There actually exist a wide literature (see e.g. Dotsey and Mao [1992]) that shows that the approximation is tiny as long as the volatility of the shocks hitting the economy is small enough. This therefore precludes the use of such technics to study big shocks, like a war, a big financial shock, or a structural change (other technics are available).

Applying this technics to the baseline RBC model, we end up with the system

$$\begin{aligned}
-\widehat{c}_t &= \widehat{\lambda}_t \\
\frac{h^\star}{1-h^\star}\widehat{h}_t &= \widehat{\lambda}_t + \widehat{w}_t \\
\widehat{y}_t &= \widehat{a}_t + \alpha\widehat{k}_t + (1-\alpha)\widehat{h}_t \\
\widehat{w}_t &= \widehat{y}_t - \widehat{h}_t \\
\widehat{z}_t &= \widehat{y}_t - \widehat{k}_t \\
\widehat{y}_t &= \frac{c^\star}{y^\star}\widehat{c}_t + \frac{i^\star}{y^\star}\widehat{i}_t \\
\gamma\widehat{k}_{t+1} &= (\gamma+\delta-1)\widehat{i}_t + (1-\delta)\widehat{k}_t \\
\widehat{a}_{t+1} &= \rho\widehat{a}_t + \varepsilon_{t+1} \\
\widehat{\lambda}_t &= \mathbb{E}_t[\widehat{\lambda}_{t+1}] + \frac{\gamma-\beta(1-\delta)}{\gamma}\mathbb{E}_t[\widehat{z}_{t+1}]
\end{aligned}$$

Parameter Values: This step of assigning values to the parameters of the model is called the *calibration* step. This literature takes this step very seriously, and it should be done very cautiously. The term *calibration* is actually borrowed from physics. Let us escape economics for a moment, and think we want to *calibrate* a thermometer. How would we do that? We would probably go to a place with altitude 0 on a normal day, such that atmospheric pressure is normal, pick up a fridge, place the thermometer in the fridge and lower the inside temperature until some pure water ices. Then we would place a 0° mark on the thermometer. Then we would pick a sauce pan, pour some water in, place the thermometer in the water, and heat it until it starts boiling. We would then tick the 100° mark on the thermometer. Then we would divide the 0–100 interval into as many uniform intervals we need to precisely determine ambient temperature.

We will do the same with the model! This approach to parameterizing the model has been introduced in economics by Prescott [1986] and was more precisely codified by Cooley and Prescott [1995]. It amounts to borrow values for the parameters from micro evidence —in particular for the utility function— and to reveal some other parameters using their implication for the long–run behavior of the model. It is important to note that we do not assign the parameters such that we make the model perfectly match some moments of interest.

Let us first focus on the parameters pertaining to technology. The parameter α is obtained from the labor share as found in the data. Indeed, when we look at the optimal behavior of the firm (assuming a Cobb–Douglas function) we get, in a steady state

$$1-\alpha = \frac{w^\star h^\star}{y^\star}$$

where the right hand side of the equation can be simply measure from the data as the ratio of labor compensation to GDP. In the case of the US economy, this share is about 60%, such that α is assigned the value 0.4.

The gross rate of growth of the economy can be simply obtained by using real GDP data.[8] On US data, the gross rate of growth of output has been 0.9% per quarter on average in the post World–war II period, such that we set $\gamma = 1.009$.

Investment is 7.6% of total physical capital stock in the US on annual data. Therefore, in a model calibrated on annual data, we would get

$$\frac{i}{k} = \gamma^y + \delta^y - 1$$

where ^{y} stands for the fact that this was calibrated on a yearly basis. Since we know γ at the quarterly frequency, we have $\gamma^y = \gamma^4$, such that

$$\delta^y = \frac{i}{k} + 1 - \gamma^y$$

which, plugging the previous numbers, yields $\delta^y = 0.0395$, which gives us a quarterly depreciation rate of $\delta = 0.0100$.

The capital stock is 3.32 times as large as GDP in the US on annual data. Then, using the Euler equation, we get

$$\beta^{y-1} = \alpha y^\star / k^\star + 1 - \delta^y$$

which leads to an annual discount factor of $\beta^Y = 0.9251$, which translates into a quarterly discount factor of $\beta = 0.9807$.

Note that looking at the log–linear system we found in the previous section, the only remaining parameter we have to assign (leaving those pertaining to shocks for the moment) is $h^\star$. Micro

[8] Note that since we did not make any distinction between population and technological progress in the model, γ is simply the rate of growth of the economy.

evidence (see e.g. Becker and G. [1975]) suggest that US households devote 31% of their total time endowment to productive activities, such that $h^\star = 0.31$. Should we be interested in θ, this value, together with that of c/y could the used to get

$$\theta = \frac{1-\alpha}{c^\star/y^\star}\frac{1-h^\star}{h^\star}$$

The last and potentially most delicate to calibrate parameters are those pertaining to the technology shock, ρ, $\overline{A}$ and σ_a. Without loss of generality, we can set $\overline{A} = 1$.[9] This implies that the process for the technology shock reduces to

$$\log(A_t) = \rho\log(A_{t-1}) + \varepsilon_t$$

It is however very difficult to have a direct measure of the shock as we do not have a technology shock in the data. This process is actually not observable. Two solutions have been proposed to circumvent the problem. The first one is to set the volatility of the shock, σ_a, so as to match the volatility of output, and to set the persistence, ρ, so as to match that of output. A second, more direct, approach builds a time series for the technology shock and directly estimates the process. It is however difficult to measure technical progress. Nevertheless, the model gives us a natural way to think about it. Indeed, in the case of perfect competition and constant returns to scale, technological progress can be measured by the Solow residual. Solow [1957] proposed to measure that part of growth that can be attributed to technological progress as output growth which is left unexplained once we accounted for growth in inputs. The solow residual, SR_t, is therefore given by:

$$\Delta\log(SR_t) = \Delta\log(Y_t) - F_k(t)\frac{K_t}{Y_t}\Delta\log(K_t) - F_h(t)\frac{h_t}{Y_t}\Delta\log(h_t)$$

Perfect competition implies

$$W_t = F_h(t) \text{ and } z_t = F_h(t)$$

such that

$$\Delta\log(SR_t) = \Delta\log(Y_t) - \frac{r_tK_t}{Y_t}\Delta\log(K_t) - \frac{W_th_t}{Y_t}\Delta\log(h_t)$$

With constant returns to scale and denoting $1-\alpha_t = \frac{W_th_t}{Y_t}$, this rewrites:

$$\Delta\log(SR_t) = \Delta\log(Y_t) - \alpha_t\Delta\log(K_t) - (1-\alpha_t)\Delta\log(h_t)$$

which, in the case of the cobb–Douglas function simplifies to

$$\Delta\log(SR_t) = \Delta\log(Y_t) - \alpha\Delta\log(K_t) - (1-\alpha)\Delta\log(h_t)$$

[9]This is because since technology exhibits constant returns to scale, the size of the economy does not matter.

Cumulating the $\Delta \log(SR_t)$, we get a time series for the Solow residual that we can regress on a linear trend

$$\log(SR_t) = \gamma_0 + \gamma_1 t + u_t$$

then the technological shock, $\log(A_t)$, corresponds to the residuals of the previous equation. The estimation of the AR(1) process yields $\rho = 0.95$ and $\sigma_a = 0.0079$.

It is however important to realize that this measure of the technological shock is highly controversial:

- It is very difficult to properly measure the physical capital stock, and it is subject to measurement errors which may bias the measure of the technology shock.
- The Solow residual can be contaminated by other shocks. Indeed, its measure rests on the assumption of *(i)* perfect competition; *(ii)* constant returns to scale and *(iii)* capital and labor are the sole inputs. These assumptions are far from being satisfied in the real world and this has important consequences. For instance, with imperfect competition the presence of markups will create a bias in the measure of the Solow residual which will imply that it will be contaminated by demand shocks. Endogenous utilization will also contaminate the technology shock…

Once we have numbers, we can set up the matrix representation of the model in order to compute the solution of the model.

Finding the solution: First of all, it is important to understand what we mean by solving the model. Solving the model actually amounts to find *decision rules* for all variables. A *decision rule* is a function that maps the information set of the agents (or a collection of exhaustive statistics) to the set of optimal choice. More precisely in this model the agent enter the period with some level of the capital stock, k_t, and are able to observe the technology shock, a_t. These two variables actually are sufficient statistics to fully characterize the state of the economy in the beginning of the period. These are therefore called *state variables*. A decision rule for consumption is therefore a function $\mathscr{C}(\cdot)$ such that $c_t = \mathscr{C}(k_t, a_t)$ is optimal for the agent. In order to get these functions, we need to solve the approximated system of equations. We saw in the first part of the course that it is possible to solve a set of linear stochastic difference equations, once we put it into the form

$$\begin{aligned} M_{yy}\mathcal{Y}_t &= M_{yx}\mathcal{X}_t \\ M_{xx0}E_t\mathcal{X}_{t+1} + M_{xx1}\mathcal{X}_t &= M_{xy0}E_t\mathcal{Y}_{t+1} + M_{xy1}\mathcal{Y}_t + M_{xe}\mathcal{E}_{t+1} \end{aligned}$$

In our case, we have

$$\mathcal{Y}_t = \begin{pmatrix} \widehat{y}_t \\ \widehat{c}_t \\ \widehat{i}_t \\ \widehat{h}_t \\ \widehat{w}_t \\ \widehat{z}_t \end{pmatrix}, \quad \mathcal{X}_t = \begin{pmatrix} \widehat{k}_t \\ \widehat{a}_t \\ \widehat{\lambda}_t \end{pmatrix}, \quad \mathcal{E}_t = \left(\widehat{\varepsilon}_t\right),$$

Then we have, for the measurement equation

$$M_{yy} = \begin{pmatrix} 1 & 0 & 0 & \alpha-1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -\frac{c^\star}{y^\star} & -\frac{i^\star}{y^\star} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{h^\star}{1-h^\star} & -1 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } M_{yx} = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

for the state equation

$$M_{xx0} = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; M_{xx1} = \begin{pmatrix} \delta-1 & 0 & 0 \\ 0 & 0-\rho & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_{xy0} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\gamma-\beta(1-\delta)}{\gamma} \end{pmatrix}, \; M_{xy1} = \begin{pmatrix} 0 & 0 & \gamma+\delta-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } M_{xe} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Using the method we presented in the first part of the course and the numbers we presented in the calibration step, we end up with the following decision rules

$$\begin{aligned}
\widehat{k}_{t+1} &= 0.9561 \; \widehat{k}_t + 0.1084 \; \widehat{a}_t \\
\widehat{a}_{t+1} &= 0.9500 \; \widehat{a}_t + \widehat{\varepsilon}_{t+1} \\
\widehat{y}_t &= 0.2418 \; \widehat{k}_t + 1.4347 \; \widehat{a}_t \\
\widehat{c}_t &= 0.6239 \; \widehat{k}_t + 0.3846 \; \widehat{a}_t \\
\widehat{i}_t &= \text{-}1.3272 \; \widehat{k}_t + 5.7463 \; \widehat{a}_t \\
\widehat{h}_t &= \text{-}0.2637 \; \widehat{k}_t + 0.7246 \; \widehat{a}_t \\
\widehat{w}_t &= 0.5055 \; \widehat{k}_t + 0.7102 \; \widehat{a}_t \\
\widehat{z}_t &= \text{-}0.7582 \; \widehat{k}_t + 1.4347 \; \widehat{a}_t
\end{aligned}$$

The decision rules accord with common intuition. More capital increases output as it raises one of the two inputs. Since more capital means higher wealth in the economy, such that the household can consume more. Investment decreases since, despite the wealth effect is positive, higher investment lowers the marginal efficiency of capital, therefore discouraging investment. The wage is higher as capital exerts a positive effect on marginal productivity of labor while the rental rate of capital decreases due to marginal decreasing returns to capital accumulation. All variables increase following a shock, as a positive shock raises output and therefore the income of the agent and at the same time exerts a positive effect on the marginal product of inputs.

4 The Model at Work

In this section, we will investigate the ability of the model to account for the Business Cycle, and we will try to locate its potential weaknesses. It is very important for you to note that the methodology is essentially quantitative: we will assess the ability of the model to match what is observed in the data.

4.1 A Successful Model

Remember that the ultimate goal of the model is to be able to mimic the business cycle, and that we characterized the business cycle by a set of statistics. Table 2 reports second order moments for the main aggregates both in the data (italic) and in the model. The moments in the model are obtained from Monte–Carlo simulations. Once we obtain a solution for the model, this works as follows

1. Draw a size $(T \times 1)$vector of innovations from a gaussian distribution with mean 0 and volatility σ_a.
2. Use the state–space solution to generate time series for the main aggregates in the model.
3. Apply the HP filter to these series
4. Compute the moments of interest (standard deviations, correlations, autocorrelations,...) and store them
5. Repeat this process N times to get a distribution of moments
6. Report the mean of this distribution

In the simulations, we used T=232 and N=1000.[10] All in all the model performs well. First of all, it generates the correct ranking of volatilities. Consumption is less volatile than output and investment is more volatile. The model predicts that the relative volatility of consumption is 0.31, it is 0.47 in the data. That of investment is 4.01 in the model, it is 3.83 in the data. One may obviously object that consumption is too smooth, but this can be very easily addressed by slightly changing preferences. For instance, using preference of the form

$$U(C_t, \ell_t) = \frac{\left(C_t^{\nu} \ell_t^{1-\nu}\right)^{1-\sigma_c} - 1}{1 - \sigma_c} \tag{21}$$

and setting $\sigma_c = 5$ enables us to exactly match a relative volatility of consumption of 0.47. This result is easily understood. By increasing σ_c, we make the intertemporal elasticity of substitution

[10] The choice of 232 is motivated by the sample size of our data set.

Table 2: HP–Filtered Moments

Variable		$\sigma(\cdot)$		$\sigma(\cdot)/\sigma(y)$		$\rho(\cdot,y)$		$\rho(\cdot,h)$		Auto(1)	
	D	M	D	M	D	M	D	M	D	M	
Output	Y	*1.70*	1.46	–	–	–	–	–	–	*0.84*	0.70
Consumption	C	*0.80*	0.46	*0.47*	0.31	*0.78*	0.89	–	–	*0.83*	0.80
Investment	I	*6.49*	5.84	*3.83*	4.01	*0.84*	0.99	–	–	*0.81*	0.70
Hours worked	h	*1.69*	0.74	*1.00*	0.51	*0.86*	0.98	–	–	*0.89*	0.69
Labor productivity	Y/h	*0.90*	0.75	*0.53*	0.51	*0.41*	0.98	*0.09*	0.92	*0.69*	0.73

Note: D stands for Data, M stands for Model.

lower — increase the size of the wealth effect — and therefore limit the possibilities of smoothing for the agent. Consumption is therefore more volatile.[11]

The model also correctly predicts that all aggregates are procyclical — meaning that they all co–vary positively with output. Finally the model displays persistence.

There are obviously some weaknesses, but let us postpone this discussion for the moment.

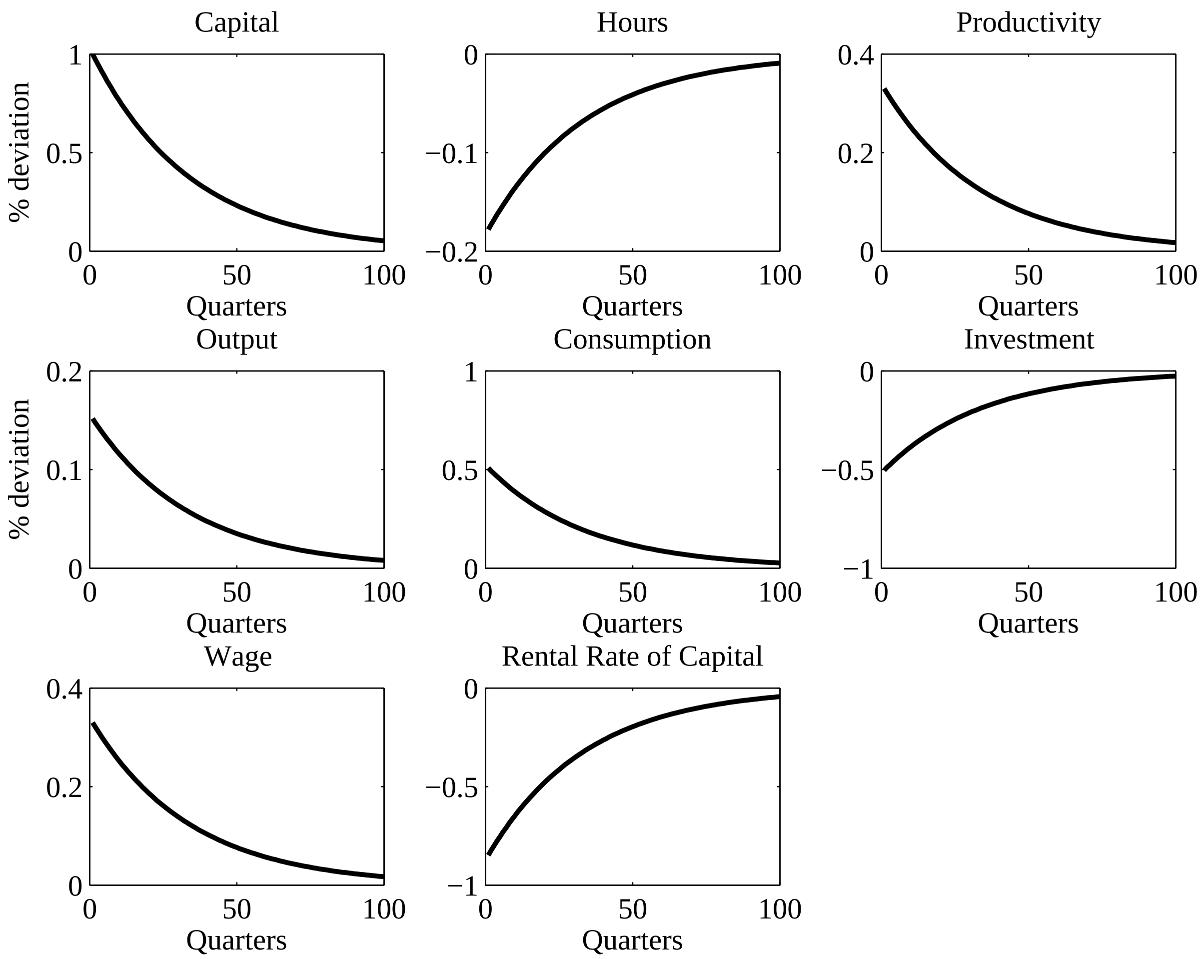
Why does it work?

To answer this question it will prove useful to investigate the dynamic properties of the model by looking at impulse response functions. As the form of the solution indicates, all decision rules depend on two state variables: capital and the technology shock. We will now investigate the role of each of them in the results.

Can it be capital accumulation? Figure 7 reports the response of the main variables to a positive one percent deviation of the capital stock from its steady state level. An increase in the capital stock amounts to have an increase in the aggregate wealth of the economy. This creates a positive wealth effect that pushes the permanent income of the agent upward. The household therefore increases his consumption. However, note that since an increase in the capital stock is perceived by the household as an increase in his permanent income, the response of consumption is quite large: 0.6% for a 1% increase in the capital stock. This therefore cannot really explain the observed excess smoothing in consumption behavior. At the same time, output increases as more capital is used in the production process. This happens despite hours drop at the same time — a phenomenon we will investigate soon. So consumption and output co–vary positively.

[11] It shall however be noticed that given the uncertainty surrounding the exact form of preferences, we do not really know the exact value of σ_c. The use of the logarithmic function is more the outcome of common practice than really theoretically grounded.

Figure 7: IRFs to a 1% increase in the Capital Stock



The increase in the capital stock seems to be able to account for the positive co–movement between output and consumption.

Because of diminishing returns to scale, the marginal productivity of capital is a decreasing function of the capital stock. The increase in the capital stock therefore implies that the rental price of capital falls (see lower right panel of the figure). Then the household faces the following trade–off. On the one hand, higher wealth enables him to invest more. On the other hand, the decrease in the rental rate of capital discourages investment. For our calibration, the second effect dominates and investment drops. The effect would reverse only if preferences are characterized by the presence of strong enough wealth effects. For instance, for the utility function (21), this would happen for values of σ_c above 5.5. In the baseline calibration, the price effect dominates, such that investment and output are negatively correlated. Therefore capital accumulation plays against the basic result. Something else is driving the procyclicality of investment.

A similar problem arises with labor decisions. Leisure is a normal good.[12] An implication of this assumption on preferences is that leisure will increase with an increase in the capital stock, and will therefore co–vary positively with consumption. But a direct implication of this result is that hours will decrease. Again, this occurs despite the increase in the wage that follows the rise in the capital stock (lower left panel of the figure), which should create incentives for the household to supply more labor. The only way to break down this result would be to kill the wealth effect in this model, which could be obtained by using preferences of the form

$$U(C_t, h_t) = \begin{cases} \frac{\left(C_t - \theta\Gamma_t \frac{h_t^{1+\sigma_h}}{1+\sigma_h}\right)^{1-\sigma_c} - 1}{1-\sigma_c} & \text{if } \sigma_c \in \mathbb{R}_+ \backslash \{1\} \\ \log\left(C_t - \theta\Gamma_t \frac{h_t^{1+\sigma_h}}{1+\sigma_h}\right) & \text{if } \sigma_c = 1 \end{cases}$$

In that case, the first order condition for the labor decision reduces to

$$\theta\Gamma_t h_t^{\sigma_h} = W_t \Longleftrightarrow \theta h_t^{\sigma_h} = w_t$$

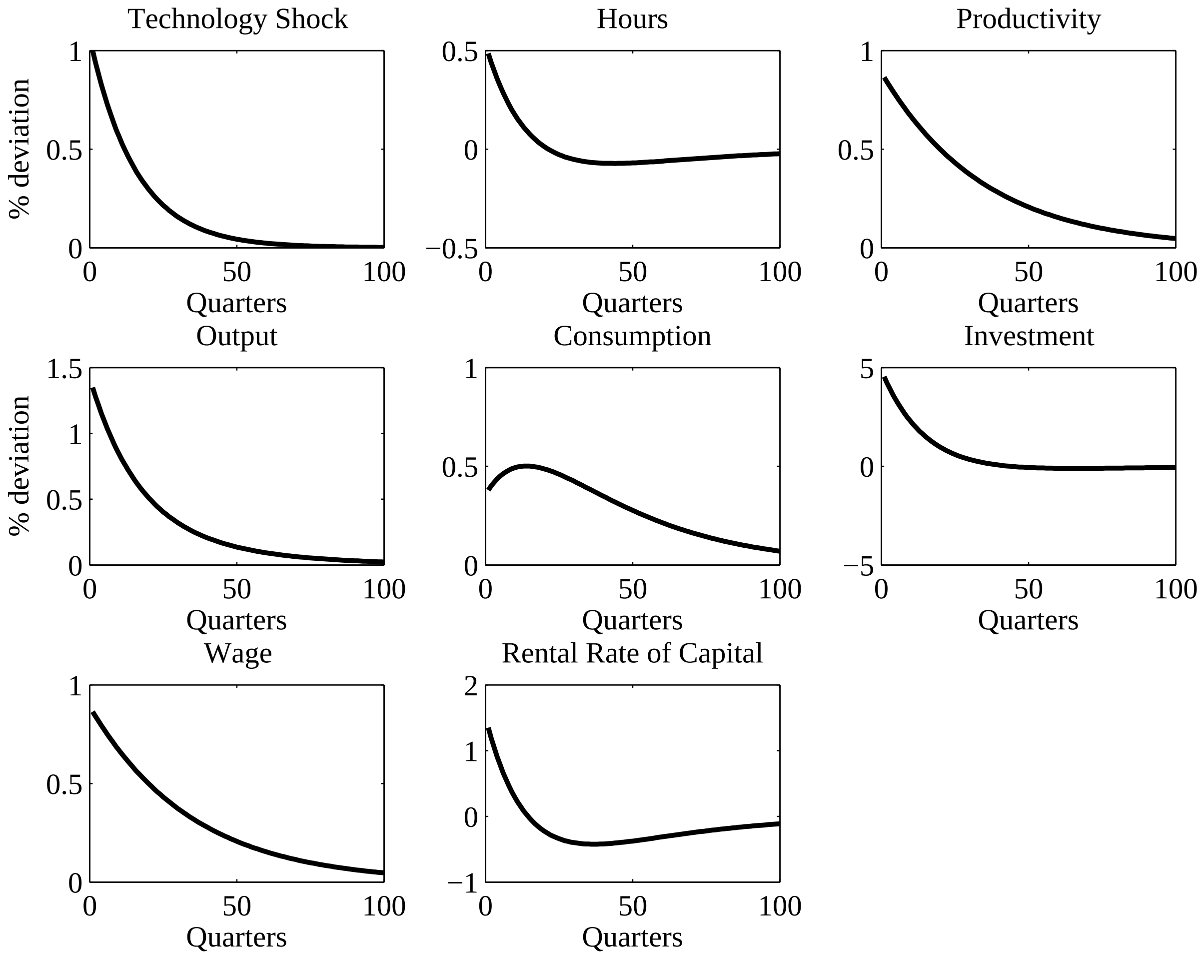
such that the increase in the capital stock, by increasing the wage would lead to higher labor supply. In other words, in presence of a wealth effect, the increase in capital leads to a drop in hours. In fact, the agent knows that he will be able to achieve the same level of consumption while enjoying more leisure. Therefore, he decreases his labor supply. Again this plays against the procyclicality of labor. The drop in hours explains why output responds less than 0.4 (the coefficient on capital) in the model. In other words, the wealth effect by itself would imply that output be less volatile than consumption, which is at odds with the empirical evidence.

Note that this does not mean that capital accumulation is a bad thing in this model. It proves very useful as the next section will make clear.

[12] Recall that a good is said to be normal if the demand for this good is increasing with wealth.

The role of productivity shocks Figure 8 reports the response of the main variables to a positive one percent technology shock.

Figure 8: IRFs to a 1% technology shock



Ceteris Paribus, the direct instantaneous of a technology shock is to raise output and the marginal product of all inputs. Therefore both the real wage and the rental rate of capital increase, and the household experiences an increase in his income. Furthermore, since this increase is persistent, as the shock displays persistence (see upper left panel in the figure), the agents perceive that their permanent income will also remain above its steady stat level for a while. This leads them to increase their consumption. This phenomenon is partially countered by the increase in the interest rate which creates a intertemporal substitution effect. Indeed, the rise in the interest rate lowers the discounted price of future consumption, which makes it beneficial for the household to postpone consumption. This can be easily seen from the budget constraint of the household which rewrites as

$$K_t = \mathbb{E}_t \left[\sum_{\tau=0}^{\infty} \left(\prod_{j=0}^{\tau} \frac{1}{z_{t+j} + 1 - \delta} \right) (C_{t+\tau} - W_{t+\tau} h_{t+\tau}) \right]$$

which can be rewritten as

$$K_t = \mathbb{E}_t \left[\sum_{\tau=0}^{\infty} q_{t+\tau} (C_{t+\tau} - W_{t+\tau} h_{t+\tau}) \right] \text{ with } q_{t+\tau} \equiv \prod_{j=0}^{\tau} \frac{1}{z_{t+j} + 1 - \delta}$$

q_t can be interpreted as the intertemporal price and is clearly a decreasing function of the future sequence of z_{t+j}. An alternative way to put it is to realize that an increase in the interest rate means that any unit of good which is invested (or saved) rather than consume immediately will yield a higher return. The household is therefore willing to consume less today and invest to benefit from higher capital income tomorrow. The positive wealth effect dominates for our calibration. Even when preferences are characterized with very high intertemporal elasticity of substitution ($\sigma_c < 0.00001$) the model still generate a positive response of consumption to a technology shock. However, as time elapses, the effects of the technology shock vanishes and the wealth effect dampens. Consumption therefore goes back to its steady state level.

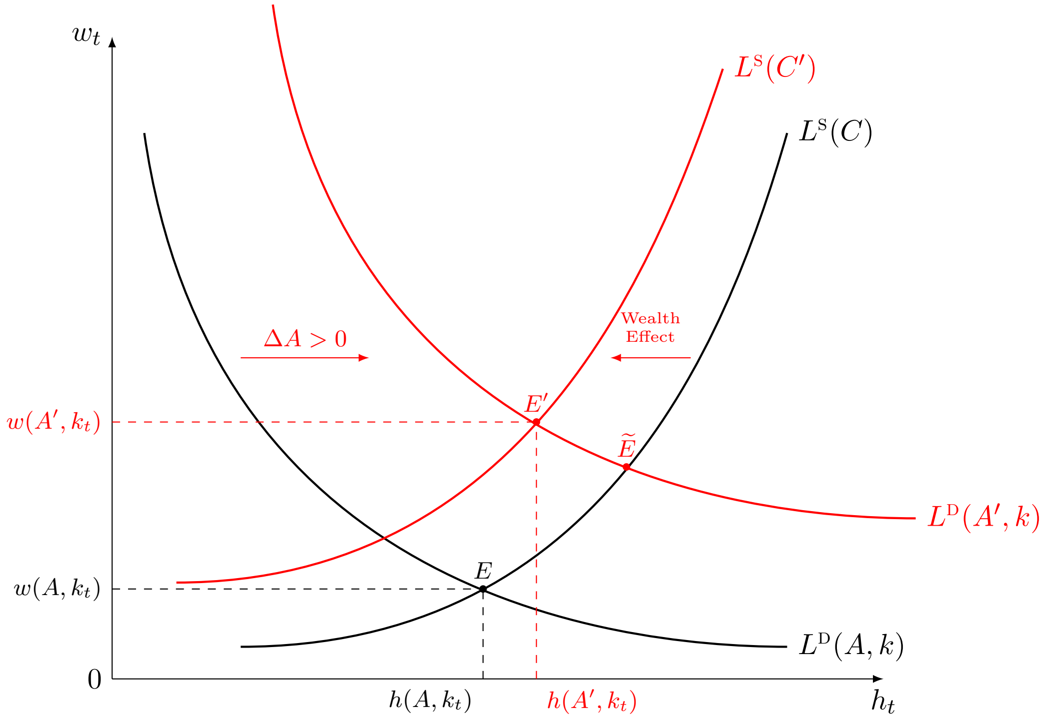
The increase in the marginal product of capital together with the positive wealth effect lead to a rise in the investment. This reinforces capital accumulation, and capital rises above its steady state level. Two mechanisms then contribute to lower interest rates. First of all, even though the technology shock is persistent, it is not permanent, and the positive effect it exerts on the marginal product of capital vanishes. Second, due to the existence of marginal diminishing returns to capital, the increase in the capital stock puts downward pressure on the marginal efficiency of capital. The interest rate therefore decreases steadily to its steady state level.

Employment dynamics is essentially explained by intertemporal substitution motives. As aforementioned, both the wage and the interest rate increase following a positive technology shock. Three main effects come into play

1. An intratemporal substitution effect;
2. An intertemporal substitution effect;
3. A negative wealth effect;

These three effects can be read in Figure 9 which depicts the functioning of the labor market. The very first effect of the technology shock is to trigger a shift of the labor demand to the right, which reflect the immediate increase in the marginal productivity of labor. This makes labor more profitable to the household which then decides to substitute labor time for leisure. This

Figure 9: Labor Market Equilibrium



is standard *intratemporal substitution effect*. On top of that, since the interest rate increases, this makes discounted future wage, $\mathbb{E}_t\left[\frac{w_{t+1}}{z_{t+1}+1-\delta}\right]$, less attractive implying that the household is willing to work more today than tomorrow. This is the *intertemporal substitution effect*. These two effects can be read on the figure as the move from E to $\widetilde{E}$. It is important to note that both effect contributes to the increase in equilibrium hours worked. However, the intertemporal substitution effect decreases as the shock vanishes. This explains why hours eventually pass below their steady state level during their adjustment to the steady state. The last effect — the intertemporal wealth effect — temporarily dominates the dynamics. The intertemporal wealth effect accounts for the fact that since agents are accumulating more, the aggregate wealth in the economy increases. Therefore the household is wealthier, which makes it possible to him to achieve the same level of consumption while having more leisure time —*i.e.* while supplying less labor. This corresponds to the left shift in the labor supply, and explains the move from $\widetilde{E}$ to E'. It is important to note that the substitution motives are dominating the dynamics in the immediate aftermaths of the shocks, such that, in the baseline RBC model, hours always increase after a persistent technology shock. It however reduces substantially the response of hours in equilibrium, therefore reflecting the existence of a smoothing effect. The effect of the technology shock together with the diminishing returns to scale imply that productivity also increase in this model.

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