Doing Economics with the Computer

Lecture 5: SOLUTIONS

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Matlab Application: Financial Evaluation 2

In this lecture we will explain how to use MATLAB to solve interest compounding problems. Borrowed from 'A Guide to MATLAB for Beginners and Experienced Users,' Brian R. Hunt, Ronald L. Lipsman, Jonathan M. Rosenberg.

Compulsory Readings

Up to now, you should have read chapters 1 - 6 in Attaway (2019). The book is provided on Ilias.

Further Exercises

Up to now, you should have solved the practice questions in chapters 1 - 6 in Attaway (2019). Solutions to the practice questions in chapter 3 - 6 are available on Ilias.

Housekeeping

When you write a script, you should include these commands at the start.

```
clear all % Deletes memory
close all % closes open windows (figures etc.)
clc       % clears command window
```

General Set-up

Consider an account that has M dollars in it and pays monthly interest J. Suppose that, beginning at a certain point, an amount S is deposited monthly and no withdrawals are made.

Definitions

- M: initial account balance in USD
- J: monthly interest rate in percent
- S: monthly deposits in USD
- T: total amount at end of period in USD
- n: the number n of months elapsed

Interest rate compounding

(a)

Assume first that S=0. Using the Mortgage Payments application as a model, derive an equation relating J, M, the number n of months elapsed, and the total T in the account after n months. Assume that the interest is credited on the last day of the month and the total T is computed on the last day after the interest is credited.

Answer:
Consider the status of the bank account on the last day of each month. At the end of the first month, the account has $M + M * J = M(1 + J)$ dollars. Then at the end of the second month the account contains $[M(1+J)](1+J) = M(1+J)^2$ dollars. Similarly, at the end of n months, the account will hold $M(1+J)^n$ dollars. So our formula is $T = M(1+J)^n$.

(b)

Now assume that M = 0, that S is deposited on the first day of the month, and that as before interest is credited on the last day of the month, and the total T is computed on the last day after the interest has been credited. Once again, using the mortgage application as a model, derive an equation relating J, S, the number n of months elapsed, and the total T in the account after n months.

Answer:
Now we take M = 0 and S dollars deposited monthly. At the end of the first month the account has $S+S*J = S(1+J)$ dollars. The next day, S dollars are added to that sum, and then at the end of the second month the account contains $[S(1+J)+S](1+J) = S[(1+J)^2 + (1+J)]$ dollars. Similarly, at the end of n months, the account will hold $S[(1+J)^n + (1+J)^{(n-1)} + ... + (1+J)]$ dollars. Summing the geometric series, the amount T in the account after n months equals $T = S[((1+J)^{(n+1)} - (J+1))/((1+J)-1)] = S[((1+J)^{(n+1)} - 1)/J - 1]$.

(c)

By combining the last two models derive an equation relating all of M, S, J, n, and T, now of course assuming that there is an initial amount in the account (M) as well as a monthly deposit (S).

Answer:
By combining the two models it is clear that, in an account with an initial balance M and monthly deposits S, the amount of money T after n months is given by $T = M(1+J)^n + S[((1+J)^{(n+1)} - 1)/J - 1]$.

(d)

If the annual interest rate is 5%, and no monthly deposits are made, how many years does it take to double your initial stash of money? What if the annual interest rate is 10%?

Answer:
We are asked to solve the equation $(1+J)^n = 2$ with the values for the monthly interest rate: $J = 0.05/12$ and $J = 0.1/12$.

```
% J = 0.05/12
syms n
months = solve((1 + 0.05/12)^n == 2);
years = double(months)/12

% J = 0.1/12
syms n
months = solve((1 + 0.1/12)^n == 2);
years = double(months)/12

% If you double the interest rate, you roughly halve the time required to achieve the goal.
```

(e)

In this and the next part, there is no initial stash. Assume an annual interest rate of 8%. How much do you have to deposit monthly to be a millionaire in 35 years (a career)?

```
format bank
1000000/(((1 + 0.08/12)^(12*35 + 1) - 1)/(0.08/12) - 1)

% You need to deposit 433.06 every month.
```

(f)

If the interest rate remains as in (e) and you can afford to deposit only 300 each month, how long do you have to work to retire a millionaire?

```
syms n
T = 300*(((1 + 0.08/12)^(n + 1) - 1)/(0.08/12) - 1);
months = solve(T - 1000000);
years = double(months)/12
```

(g)

You hit the lottery and win 100,000. You have two choices: take the money, pay the taxes, and invest whats left; or receive 100,000/240 monthly for 20 years, depositing whats left after taxes. Assume that a 100,000 windfall costs you 35,000 in federal and state taxes, but that the smaller monthly payoff causes only a 20% tax liability. In which way are you better off 20 years later? Assume a 5% annual interest rate here.
Note that this is exactly the problem one faces going into pension in CH with regard to the choice of what to do with the accumulated second layer funds.

g) First Option

First, taking all the money at once and saving it, after 20 years the 65,000 left after taxes generate:

```
option1 = (100000-35000)*(1 + 0.05/12)^(12*20)

% The stash grows to about 176,322.
```

g) Second Option

The second option yields

```
S = 0.8*(100000/240);
option2 = S*(((1 + 0.05/12)^(12*20 + 1) - 1)/(0.05/12) - 1)

% You accumulate only 137,582 this way.

% Taking the lump sum up front is clearly the better strategy.
```

(h)

Historically, banks have paid roughly 5%, while the stock market has tended to return 8% on average over a 10-year period. So parts (e) and (f) relate more to investing than to saving. But suppose that the market in a 5-year period returns 13%, 15%, -3%, 5% and 10% in five successive years, and then repeats the cycle. (Note that the [arithmetic] average is 8%, though a geometric mean would be more relevant here.) Assume that 50,000 is invested at the start of a 5-year market period. How much does it grow to in 5 years? Now recompute four more times, assuming that you enter the cycle at the beginning of the second year, the third year, etc. Which choice yields the best/worst results? Can you explain why? Compare the results with a fixed-rate account paying 8%. Assume simple annual interest. Redo the five investment computations, assuming that 10,000 is invested at the start of each year. Again analyze the results.

h.i)

```
rates = [.13, .15, -.03, .05, .10, .13, .15, -.03, .05 0.1];
for k = 0:4
    T = 50000;
    for j = 1:5
        T = T*(1 + rates(k + j));
    end
    disp([k + 1, T])
end
```

Note on the result

The results are all the same, you wind up with 72,794 USD regardless of hwere you enter in the cycle because the product $\prod_{1\leq j\leq 5}(1+\text{rates(j)})$ is independent of the order in which you place the factors. If you put the 50,000 in a bank account paying 8%, you get $50000 * (1 + 0.08)^5 = 73,466$ which is better than the market. The market's volatility hurts you compared with the bank's stability. But of course that assumes you can find a bank that will pay 8%.

h.ii)

Now let's see what happens with no stash, but an annual investment instead. The analysis is more subtle here. Set S = 10,000. At the end of one year, the account contains $S(1 + r_1)$; then at the end of the second year $[S(1+r_1)+S](1+r_2)$, where we have written r_j for rates(j). So at the end of five years, the amount in the account will be the product of S and the number $\prod_{j\geq 1}(1+r_j)+\prod_{j\geq 2}(1+r_j)+\prod_{j\geq 3}(1+r_j)+\prod_{j\geq 4}(1+r_j)+(1+r_5)$ If you enter at a different year in the business cycle the terms get cycled appropriately. So now we can compute

```
for k = 0:4
    T = ones(1, 5);
    for j = 1:5
        TT = 1;
        for l = j:5
            TT = TT*(1 + rates(k + l));
        end
        T(j) = TT;
    end
    disp([k + 1, 10000*sum(T)])
end
```

Note on the result

Not surprisingly, all the amounts are less than what one obtains by investing the original 50,000 all at once. But in this model it matters where you enter the business cycle. Its clearly best to start your investment program when a recession is in force and end in a boom. Incidentally, the bank model yields in this case $10000 * (((1 + 0.08)^6 - 1)/0.08 - 1) = 63359.290$ which is better than some investment models and worse than others.

Additional Exercises

(a)

Assume now that M=1000, S = 0. Assume again that the interest is credited on the last day of the month and the total T is computed on the last day after the interest is credited. Let the monthly interest rate be generated randomly, following a uniform distribution on the interval (0.003, 0.005).

```
a=0.003;
b=0.005;
J = a + (b-a).*rand(1,1); % J: monthly interest rate

M = 1000;                  % M: initial account balance
S = 0;                     % S: monthly deposits
n = 60;

T = M*(1+J).^n;

fprintf('Total amount in the account: %.3f USD \n',T);
```

(b)

Now assume that M = 0, S=50, that S is deposited on the first day of the month, and that, as before, interest is credited on the last day of the month, and the total T is computed on the last day after the interest has been credited. Compute the total T in the account after n=60 months.

(b) Method 1

```
M = 0;
S = 50;
n = 60;

T = S* (1 + J)*(((1+J)^(n)-1)/J) % Standard Formula
fprintf('Total amount in the account: %.3f USD \n',T);
```

(b) Method 2

```
T = M*(1 + J)^n + S*(((1 + J)^(n+1) - 1)/J - 1) % Standard Formula
fprintf('Total amount in the account: %.3f USD \n',T);
```

(b) Method 3

At the end of 1, 2, 3,.. period we have an amount: $1 : 50 * (1 + r)$
$2 : 50 * (1 + r)^2 + 50 * (1 + r)$
$3 : 50 * (1 + r)^3 + 50 * (1 + r)^2 + 50 * (1 + r); ...$
Thus after n periods we have
$50 * (1 + r) * [(1 + r)^{n-1} + (1 + r)^{n-2} + ... + 1]$.
The term inside [] is a geometric series that starts with 1 and grows at the rate of $1 + r$. The formula above corresponds to the sum in [...] Alternatively, note that

$T(2) = T(1) + (1+r)^2$;
$T(3) = T(2) + (1+r)^3, ..., T(n+1) = T(n) + S*(1+r)^n$
and use the familiar for loop routine where $T(1) = 50*(1+r)$.

```
T(1)=S*(1+J);

for i = 2:n
    T(i)= S*(1+J)^i + T(i-1);
end

fprintf('Total amount in the account: %.3f USD \n',T(end));
```

(c)

Let $M = 1000, S = 50$ and compute the total amount T in the account after $n = 60$ months. Note that at the end of period 1, 2, 3, .., T, .. we have an amount $T(1) = 1000*(1+r) + 50*(1+r)$;
$T(2) = 1000*(1+r)^2 + 50*(1+r)^2 + 50(1+r)$;
$T(3) = 1000*(1+r)^3 + 50*(1+r)^3 + 50*(1+r)^2 + 50*(1+r)$;
$T(4) = 1000*(1+r)^4 + 50*(1+r)^4 + 50*(1+r)^3 + 50*(1+r)^2 + 50*(1+r)$;
$T(4) - T(3) = 1000*(1+r)^4 + 50*(1+r)^4 - 1000*(1+r)^3 = 1000*(1+r)^3(1+r-1) + 50*(1+r)^4$
$\rightarrow T(4) = T(3) + 1000*(1+r)^3(1+r-1) + 50*(1+r)^4$;
$T(n) = 1000*(1+r)^n + 50*(1+r)^n + 50*(1+r)^{n-1} + 50*(1+r)^{n-2} + .. + 50*(1+r)$;
Hence:
$T(n+1) = T(n) + 1000*(1+r)^{(n-1)}*r + 50*(1+r)^n$

```
a=0.003;
b=0.005;
J = a + (b-a).*rand(1,1); % J: monthly interest rate

M = 1000;
S = 50;
n = 60;
```

Method 1

```
T = M*(1 + J)^n + S*(((1 + J)^(n+1) - 1)/J - 1); % Standard Formula
fprintf('Total amount in the account: %.3f USD \n',T);
```

Method 2

```
T(1)= 1000*(1+J) +50*(1+J);

for i=2:n
    T(i) = 1000*((1+J)^(i-1))*J +50*(1+J)^i + T(i-1);
end
fprintf('Total amount in the account: %.3f USD \n',T(end));
```