Session: Portfolio Selection in a Nutshell

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1 The Topic

The financial markets of developed economies provide countless opportunities to investors. As a consequence, when making an investment we have to decide what amount we want to spend on which asset, i.e. we have to select the mixture of assets (a so called *portfolio*) which is optimal for us. Optimality thereby usually means that we wish to earn the highest possible amount of money at the lowest possible risk. In other words: given a certain return we want to minimize the corresponding risk, or given a specific risk we want to earn the maximum return possible.

2 Definitions and Statistical Concepts: Portfolio

A portfolio is a blend of various assets. In other words, when putting together a portfolio, we attach a weight $w_i \in [0,1]$ to each asset i (Note that we do not allow for negative weights, i.e. short positions.) Of course the sum of these weights has to add up to 1, i.e. $\sum_{i=1}^{N} w_i = 1$. Given our assumption that the returns of all assets R are normally distributed, we can then calculate the expected return and risk for the portfolio. [1]

For the expected return of the portfolio, μ_P, the computation is rather easy (provided that we know the expected returns and corresponding variance, μ and Σ, and w).

$$\mu_P = \sum_{i=1}^{n} w_i \mu_i \tag{1}$$

w_i is the weight of asset i in the portfolio, μ_i is the expected return of asset i and n is the total no. of assets in the portfolio.

For risk, however, things are a little more complicated. The reason why

*Slightly rearranged by Guido Baldi

[1] Doing so, we have to take into account the following fact from the theory of the multivariate normal distribution:(see e.g. Hogg et al. (2005), p. 173.)
Suppose that the returns of the n assets, R, have the distribution $R \sim N_n(\mu, \Sigma)$ (where μ is the $n \times 1$ vector of means and Σ is the $n \times n$ variance-covariance matrix.) Now let $P = wR$ where w is a $1 \times n$ vector of weights, i.e. $w = (w_1 \ \dots \ w_n)$. Then $P \sim N(w\mu, w\Sigma w')$, where $w\mu$ and $w\Sigma w'$ are scalars.
For the present case this means that if the returns have a multivariate normal distribution, the portfolio P has a univariate normal distribution with mean $w\mu$ and variance-covariance matrix $w\Sigma w'$.

is that, in general, the variance of a sum is not the sum of the variances; only if summands are not correlated with each other, this will be the case. Thus, for a portfolio of assets, variance σ_P^2 and standard deviation σ_P are generally defined as

$$\sigma_P^2 = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=1, i \neq j}^{n} w_i \cdot w_j \cdot \sigma_{ij} \tag{2}$$

$$\sigma_P = \sqrt{\sigma_P^2} \tag{3}$$

where σ_i^2 is the variance of the expected return of asset i and σ_{ij} is the covariance between the return of assets i and j.

Note that we could have stated equation 2 equivalently as

$$\sigma_P^2 = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=1, i \neq j}^{n} w_i \cdot w_j \cdot \rho_{ij} \cdot \sigma_i \cdot \sigma_j$$

where ρ_{ij} is the correlation coefficient between assets i and j. This equivalence stems from the fact that ρ_{ij} is defined as

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \cdot \sigma_j} \tag{4}$$

Examples:

For a portfolio of 2 assets, expected return μ_P and risk σ_P (defined as standard deviation) are

$$\mu_P = w_1 \mu_1 + w_2 \mu_2 \tag{5}$$

$$\begin{aligned} \sigma_P &= \left(w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \sigma_{12} \right)^{\frac{1}{2}} \qquad (6) \\ &= \left(w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \sigma_1 \sigma_2 \rho_{12} \right)^{\frac{1}{2}} \end{aligned}$$

For a portfolio of 3 assets, we have

$$\mu_P = w_1 \mu_1 + w_2 \mu_2 + w_3 \mu_3 \tag{7}$$

$$\begin{aligned} \sigma_P &= \left(w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + w_3^2 \sigma_3^2 + \ldots \right. \qquad (8) \\ &\qquad \left. \cdots + 2 w_1 w_2 \sigma_{12} + 2 w_1 w_3 \sigma_{13} + 2 w_2 w_3 \sigma_{23} \right)^{\frac{1}{2}} \\ &= \left(w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + w_3^2 \sigma_3^2 + \ldots \right. \\ &\qquad \left. \cdots + 2 w_1 w_2 \sigma_1 \sigma_2 \rho_{12} + 2 w_1 w_3 \sigma_1 \sigma_3 \rho_{13} + 2 w_2 w_3 \sigma_2 \sigma_3 \rho_{23} \right)^{\frac{1}{2}} \end{aligned}$$

3 References

Loderer, C., P. Jörg, K. Pichler, L. Roth and P. Zgraggen (2002) *Handbuch der Bewertung*. 2. erweiterte Auflage. Zürich: Verlag NZZ

Hogg, R.V., J.W. McKean and A.T. Craig (2005) *Introduction to Mathematical Statistics*. Sixth edition. Upper Saddle River, NJ: Pearson Prentice Hall

4 Task

Exercises

You are given the following data on return, risk (i.e. standard deviation) and correlations of the assets A, B and C:

	A	B	C
Return	22.5000%	11.0000%	18.1000%
Risk	22.16416%	17.578396%	15.1225%

Correlations	A	B	C
A	1.0000	-0.4787	-0.7107
B	-0.4787	1.0000	-0.2774
C	-0.7107	-0.2774	1.0000

1. Which asset is inefficient? It might be useful to plot the three assets in a graph in order to determine this.

2. Compute the covariance matrix for the three assets, using equation (4).

3. By varying the weights of A and C in a portfolio that only consists of these two assets, find the corresponding values for return and risk. You may want to do so, by writing a table of the form

Weight A	Weight C	Return	Risk
1.0	0.0	...%	...%
0.9	0.1	...%	...%
$\vdots$	$\vdots$	$\vdots$	$\vdots$

   Apply equation (5) in the third- and equation (6) in the fourth column. Make sure that the weights sum up to 1.

4. Plot the risk-return combinations that you obtained in exercise 3. What do you find?

5. Compute the minimum risk portfolio by varying the weights for A and C. Again, use equations (5) and (6) for return and risk. In carrying out the minimization, you might want to use Excel solver. Take the following constraints into account: The weights of both assets must be larger than 0 and smaller than 1. In addition, the weights have to sum up to 1.

6. Carry out the same minimization, but this time also include asset B in the portfolio. Consequently, equations (7) and (8) are needed to compute return and risk. Compare the resulting return and risk to the minimum risk portfolio you found in 5.