# Session 3: Computational Game Theory 

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## 1 Introduction

In the first part of this session we will look at the use of computers for solving problems in game theory. We don't expect you to have previous knowledge of game theory, concepts necessary for this session will be explained in class. However, it might be helpful to take a brief look at the definition of a Nash equilibrium in a microeconomics textbook. (For a simple definition look at the chapter on game theory in Varian [?]; for a more general definition Kreps [?] p. 404).
With two players A and B "a pair of strategies is a Nash equilibrium if A's choice is optimal, given B's choice, and B's choice is optimal given A's choice." (Varian [?], p. 482) i.e. a pair of strategies is a Nash equilibrium if neither of the players has an incentive to deviate from his actual strategy, given the strategy of the other player.

Two basic types of Nash equilibria are the Nash equilibrium in pure strategies, and the Nash equilibrium in mixed strategies. At a Nash equilibrium in mixed strategies, the players choose the probabilities with which they play their strategies, such that no player has an incentive to alter his choice of probabilities, given the choice of probabilities of the other player. We will consider two examples of games that yield a Nash equilibrium in mixed strategies (football game and coordination game).

## 2 Football Game

Consider the following game: a striker has passed the defense and only the goalkeeper is left between him and the goal. He has two possibilities: shoot the ball to the left or to the right. The goalkeeper can either jump to the left (from the striker's point of view) or to the right. If the goalkeeper jumps into the same direction as the striker shoots the ball, he can catch it. Otherwise, the striker scores. The payoffs of the game are:


What do you think are the optimal strategies of the two players?

### 2.1 Exercise: Simulation

Assume that the goalkeeper jumps to the left with probability 0.4. Interesting questions then are: With what probability should the striker shoot to the left? What is the payoff of the striker if he shoots to the left with probability $x$ ?
Use Excel to calculate the striker's average payoff for 1000 random draws, for different values of $x$. That is, generate 1000 random draws for the behavior of the goalkeeper, given that he jumps left with a probability of 0.4 . Then generate 1000 random draws for the behavior of the striker, given that he shoots left with probability $x$. Match the outcomes of the random
draws and compute the average payoff per shot of the striker. Vary $x$. What value of $x$ should the striker choose, given the probability of the goalkeeper to jump left?

Hint: To create a random number which is uniformly distributed between 0 and 1 in the German version of Excel enter "=ZUFALLSZAHL()" If you want to have "left" in a cell with probability 0.3 and "right" with probability 0.7 type "=WENN (ZUFALLSZAHL()<0.3;"left"; "right")" ${ }^{2}$.
Run the simulation for different values for the probability of the goalkeeper jumping to the left (e.g. 0, 0.5, 0.6, 1).

### 2.2 Exercise: Find the Equilibrium

Create an $11 \times 11$ table with the columns as the probability of the striker shooting to the left ( $p_{\text {shoot left }}=0,0.1,0.2, \ldots 1$ ) and the rows as the probabilities of the goalkeeper of jumping left ( $p_{\text {jump left }}=0,0.1,0.2, \ldots 1$ ). Fill the table with the expected payoffs of the striker for the specific $p_{\text {shoot left }}$ and $p_{\text {jump left }}$. Do you see points where the striker has no interest in deviating unilaterally?
Do the same for the goalkeeper.
Is there a Nash equilibrium in mixed strategies?

## 3 Coordination Game

A couple loses each other in a shopping mall and tries to find each other again. The shopping mall has a west exit and an east exit. If they both go to the same exit, they can go home together (payoff 1), otherwise they have to continue searching for each other for hours (payoff $0)$.


What strategy should they pursue?

### 3.1 Exercise: Simulation

Where should one go if one's partner goes to the west exit with probability 0.4 ? Try different probabilities and calculate the average payoff for 1000 random draws as in 2.1. In what respect do your findings differ from those derived in 2.1?

### 3.2 Exercise: Find the Equilibrium

Create a table as in 2.2. Note that only one table is necessary, as both players have the same payoff structure. For each player mark the points where he/she has no incentive to

[^1]unilaterally deviate.
Is there a Nash equilibrium in mixed strategies?
Compare your results to those derived from 2.2.

### 3.3 Algorithmic Solution

Using Excel Solver, there is a procedure that is much faster than the one applied in exercises 2.2 and 3.2. McKelvey (1992) [?] has developed an algorithm to find the mixed strategy (i.e. probabilistic) Nash equilibria of a game numerically. It is a general algorithm for an arbitrary number of players with an arbitrary number of strategies, but we will look for the specific algorithm for the coordination game.
We define the function $v(\cdot)$ as ${ }^{3}$

$$
v\left(p_{1}^{1}, p_{2}^{1}, p_{1}^{2}, p_{2}^{2}\right)=\sum_{i=1}^{2} \sum_{j=1}^{2}\left[\max \left\{0, p_{j}^{i}-\left(p_{1}^{1} p_{1}^{2}+p_{2}^{1} p_{2}^{2}\right)\right\}\right]^{2}
$$

with $p_{j}^{i}$ denoting the probability that player $i$ plays strategy $j$. (Player 1 is the man, player 2 the woman, strategy 1 is the west exit, strategy 2 the east exit).
All Nash equilibria of the game are minima of $v(\cdot)$, further $v()$ has the value 0 for a Nash equilibrium.
To find the Nash equilibria all we have to do is tell the Excel Solver to minimize $v()$.
Exercise:
Create an Excel sheet with some initial value for $p_{1}^{1}$ and $p_{1}^{2}$. (Because the probabilities of the strategies of a player have to sum up to 1 , you can express $p_{2}^{1}$ as $\left(1-p_{1}^{1}\right)$ and $p_{2}^{2}$ as $\left(1-p_{1}^{2}\right)$.) Enter the $v()$ function. In the Excel menu choose "Extras:Solver" ${ }^{4}$ and the dialog depicted in fig. 1 should appear. As the target cell ("Zielzelle") choose the cell where $v()$ is. Choose to minimize the target value ("Zielwert: Min"). Choose the cells with the probabilities of going to the west exit and the east exit as variable cells ("Veränderbare Zellen"). As constraints ("Nebenbedingungen") enter that probabilities have to be between 0 and 1.
Now click on the solve button ("Lösen"). Excel will try varying the probabilities in order to minimize $v()$.

What equilibrium do you get?
Try different starting values for the probabilities and rerun the solver. What equilibria do you get?
Enter $p_{1}^{1}=0.25$ and $p_{1}^{2}=0.25$ and run the Solver. What do you get? Is this a Nash equilibrium?

[^2]

Figure 1: Solver Dialog

## 4 Appendix (Optional): More General McKelvey Algorithm

If there is enough time left, you can look at a more general version of the McKelvey algorithm. Here we will look at the solution of a two player two strategies game (see Judd (1998) [?] for an even more general version of the algorithm). Consider the following game:

> |  |  | Player 2 |  |
| :---: | :---: | :---: | :---: |
| Player 1 | strategy 1 |  |  |
|  | strategy 2 |  |  |
|  | $m_{1,1}^{1}, m_{1,1}^{2}$ | $m_{1,2}^{1}, m_{1,2}^{2}$ |  |
|  | $m_{2,1}^{1}, m_{2,1}^{2}$ | $m_{2,2}^{1}, m_{2,2}^{2}$ |  |

Here $m_{j, k}^{i}$ is the payoff of player $i$ if player 1 plays $j$ and player 2 plays $k$. Let $p_{j}^{i}$ still denote the probability that player $i$ plays strategy $j$.
Now function $v()$ has the following value:

$$
\begin{align*}
v\left(p_{1}^{1}, p_{2}^{1}, p_{1}^{2}, p_{2}^{2}\right)= & \sum_{j=1}^{2}\left[\max \left\{p_{1}^{2} m_{j, 1}^{1}+p_{2}^{2} m_{j, 2}^{1}-\sum_{k=1}^{2} \sum_{l=1}^{2} p_{k}^{1} p_{l}^{2} m_{k, l}^{1}\right\}\right]^{2} \\
& +\sum_{j=1}^{2}\left[\max \left\{p_{1}^{1} m_{1, j}^{2}+p_{2}^{1} m_{2, j}^{2}-\sum_{k=1}^{2} \sum_{l=1}^{2} p_{k}^{1} p_{l}^{2} m_{k, l}^{2}\right\}\right]^{2} . \tag{1}
\end{align*}
$$


[^0]:    *Slightly rearranged by Marco Riguzzi and Guido Baldi

[^1]:    1 " $=$ RAND ()" in the English version
    ${ }^{2}=\operatorname{IF}($ RAND ()$<0.3, " l e f t ", " r i g h t ")$

[^2]:    ${ }^{3}$ This function is specific to the payoffs of the coordination game. Equation (1) is the general form for a $2 \times 2$ game.
    ${ }^{4}$ in English version: Tools:Solver; in Excel 2007: Daten: Solver

