Session 2: Hog Cycle Model

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1 The Topic

The Hog Cycle or Cobweb Model is a standard microeconomic supply and demand model for a single good with a dynamic extension.[1] It is called Hog[2] Cycle Model, because the supply of pork in the US and its price shows the features of this model.

Since only one commodity is being considered, it is necessary to include only three variables in the model: the quantity demanded of the commodity, QD_t, the quantity supplied of the commodity, QS_t, and its price, P_t, The quantity is measured, say, in tons per week, and the price in dollars. We make certain assumptions:

First, we must specify an equilibrium condition. The standard assumption is that equilibrium obtains in the market if and only if excess demand is zero ($QD_t - QS_t = 0$). Second we must describe how QD_t and QS_t itself are determined. We assume that QD_t is a decreasing linear function of P_t. QS_t is postulated to be an increasing linear function of the expected value of P_t, based on information available in $t-1$, with the proviso that no quantity is supplied unless the price exceeds a particular positive level. That is, we assume that the suppliers output decision must be made one period in advance of the actual sale - such as in agricultural production, where planting must precede by an appreciable length of time the harvesting and sale of the output. Third, we assume that farmers form their expectations in a naive fashion: they base their decision in period $t-1$ for their supply in t, QS_t , simply on the observed price in period $t-1$, P_{t-1}.

Translated into mathematical statements, the model can be written as

$$QD_t = QS_t \tag{1}$$

$$QD_t = a - bP_t, \quad (a, b > 0) \tag{2}$$

$$QS_t = -c + dE_{t-1}(P_t), \quad (c, d > 0) \tag{3}$$

$$E_{t-1}(P_t) = P_{t-1} \tag{4}$$

Four exogenous parameters a, b, c, and d appear in the two linear functions, and all of them are specified to be positive.

Equations (1) to (4) can be simplified as follows. We know that the solution values of the three endogenous variables, QD_t, QS_t, and P_t are those values that satisfy the three equations (1) to (3) simultaneously. Hence, by substituting the last three equations into the first the model can be reduced

*Slightly rearranged by Jürg Adamek and Guido Baldi

[1] As we shall see below, the static theory of price and quantity determination in an isolated market is modified by introducing a time delay in the supply function.

[2] german: Hausschwein.

to a single *linear first-order difference equation*

$$bP_t + dP_{t-1} = a + c \tag{5}$$

or, equivalently,

$$P_t = \frac{-d}{b}P_{t-1} + \frac{(a+c)}{b} \tag{6}$$

In the next section we shall ask ourselves how a dynamic system like (6) evolves over time and what the effects on P_t of changes in the parameter values a, b, c and d are.

2 The Method

The purpose of this section is to provide you with the basics of a mathematical tool called *difference equations* which is widely used in econometrics and macroeconomics.[3]

2.1 Linear First-Order Difference Equations

We say a variable y_t follows a *linear first-order difference equation* if its dynamics can be described by

$$y_t = \phi y_{t-1} + w_t, \tag{7}$$

where ϕ is a parameter and w_t is an exogenous function of time. In other words, equation (7) relates the variable y_t to its previous value and a so-called driving force. This is a first-order difference equation because only the first lag of the variable (y_{t-1}) appears in the equation. It is linear because no y term (of any period) is raised to power or is multiplied by a y term of another period. If the term w_t is nonzero the above difference equation is called *nonhomogenous*.

[3] A related topic are *differential* equations. Both difference and differential equations describe the evolution of a variable(s) over time, t, the former in discrete time (t = 1, 2, 3 ...) and the latter in continuous time.

2.2 Time Path of y

How does a dynamic system of that kind evolve over time? For each date t we have an equation relating the value of y for that date to its previous value and the current value of w:

Date	Equation
0	$y_0 = \phi y_{-1} + w_0$
1	$y_1 = \phi y_0 + w_1$
2	$y_2 = \phi y_1 + w_2$
$\vdots$	$\vdots$
T	$y_t = \phi y_{t-1} + w_t$

Suppose we know the starting value of y for date $t = -1$ and the value of w for dates t = 0, 1, 2, ... Hence, it is possible to calculate the value of y for any date. For example, if we know the value of y for $t = -1$ and the value of w for $t = 0$, we can calculate the value of y for $t = 0$ directly from $y_0 = \phi y_{-1} + w_0$. Given this value of y_0 and the value of w_1, we can calculate the value for y_1, and so on.

3 Solution

A solution to a difference equation expresses the value y_t as a mathematical function of the (known) elements w_t and some given initial condition of y_0.

Let's see how to solve (7).

Given the value of y_0 and the value of w_1 we can calculate:

$$\begin{aligned} y_1 &= \phi y_0 + w_1 \\ y_1 &= \phi(\phi y_{-1} + w_0) + w_1 \\ y_1 &= \phi^2 y_{-1} + \phi w_0 + w_1 \end{aligned}$$

Given this value of y_1 and the value of w for $t = 2$, we can calculate the value of y for $t = 2$:

$$\begin{aligned} y_2 &= \phi y_1 + w_2 \\ y_2 &= \phi(\phi^2 y_{-1} + \phi w_0 + w_1) + w_2 \\ y_2 &= \phi^3 y_{-1} + \phi^2 w_0 + \phi w_1 + w_2 \end{aligned}$$

Continuing recursively in this fashion, the value that y takes on at date t can be described as a function of its initial value $y - 1$ and the history of w between date 0 and date t:

$$\begin{aligned} y_t &= \phi^{t+1} y_{-1} + \phi^t w_0 + \phi^{t-1} w_1 + \phi^{t-2} w_2 + \ldots + \phi w_{t-1} + w_t \\ y_t &= \phi^{t+1} y_{-1} + \sum_{j=0}^{t} \phi^j w_{t-j} \end{aligned} \tag{8}$$

Of course (8) can alternatively be written as

$$y_{t+j} = \phi^{j+1} y_{t-1} + \phi^j w_t + \phi^{j-1} w_{t+1} + \phi^{j-2} w_{t+2} + \ldots + \phi w_{t+j-1} + w_{t+j} \tag{9}$$

This procedure is known as solving the difference equation (7) by recursive substitution. There are alternative methods. Those who are interested in that issue should consult one of the textbooks mentioned in References.

3.1 Dynamic Stability

Let's look at the solution in (8). Suppose $w_t = w$ for all t and y_{-1} is an arbitrary constant. Given sufficient time for the adjustment process, does our dynamic system tend to converge to a particular equilibrium value, $\overline{y}$, or does it not?[4] (8) becomes to

$$\lim_{t\to\infty} y_t = \phi^{t+1} y_{-1} + w\left[\phi^t + \phi^{t-1} + \phi^{t-2} + \ldots + \phi + 1\right]$$

Evidently, the answer crucially depends on the parameter ϕ. *Dynamic stability* is achieved if and only if $|\phi| < 1$. In this case (8) can be written as

$$\lim_{t\to\infty} y_t = \frac{1}{1-\phi} w = \overline{y}$$

In general, there are six cases of dynamics: the stable monotone case, the stable oscillatory case, the unstable monotone case, the unstable oscillatory case, and two borderline cases. Borderline case 1 lies between the stable and the unstable monotone case. In this case, the output variable y is the sum of the historical inputs w plus the initial value of y, y_{-1}. Correspondingly boundary case 2 lies between the stable and the unstable oscillatory case.

3.2 Effects on y of changes in w

The fact that (8) expresses y_t as a linear function of the initial value y_{-1} and the historical values of w makes it very easy to calculate the effect of

[4] Formally, the linear difference equation $y_t = \phi y_{t-1} + w_t$ is called *stable* if, when $w_t = w$ for all t and y_{-1} is an arbitrary constant, then

$$\lim_{t\to\infty} y_t = \overline{y}$$

where $\overline{y}$ does not depend on y_{-1}.

w_0 on y_t. If w_0 were to increase by one unit with y_{-1} and w_1, w_2, ..., w_t taken as unaffected (we call this a *transitory* change in w), the effect on y_t would be given by

$$\frac{\partial y_t}{\partial w_0} = \phi^t \tag{10}$$

or, alternatively,

$$\frac{\partial y_{t+j}}{\partial w_t} = \phi^j \tag{11}$$

Thus the so-called *impulse-response function* in (11) depends only on j, the length of time separating the disturbance to the input (w_t) and the observed value of the output (y_{t+j}).

Again different values of ϕ in (7) can produce a variety of dynamic responses of y to w. If $0 < \phi < 1$, the impulse-response function decays geometrically toward zero the farther into the future one goes. If $-1 < \phi < 0$, the impulse-response function will alternate in signs. In this case an increase in w_t will cause y_t to be higher, y_{t+1} to be lower, y_{t+2} to be higher, and so on. The absolute value of the effect of a given change in w_t eventually dies out. If $\phi > 1$, the impulse-response function increases exponentially over time. A given increase in w_t has a larger effect the farther into the future one goes. For $\phi < -1$, the system exhibits explosive oscillation. In the two borderline cases the initial effect remains in action for ever.

So far, we were looking for the effect of a purely transitory change in w. However, sometimes we might be interested in the consequences of a *permanent change* in w. A permanent change in w means that w_t, w_{t+1}, ..., and w_{t+j} would all increase by one unit. From formula (11), the effect on y_{t+j} of a permanent change in w beginning in period t is given by

$$\frac{\partial y_{t+j}}{\partial w_t} + \frac{\partial y_{t+j}}{\partial w_{t+1}} + \frac{\partial y_{t+j}}{\partial w_{t+2}} + \ldots + \frac{\partial y_{t+j}}{\partial w_{t+j}} = \phi^j + \phi^{j-1} + \phi^{j-2} + \ldots + \phi + 1$$

When $|\phi| < 1$, the limit of this expression as j goes to infinity, is sometimes described as the "long run" effect of w on y:

$$\lim_{j\to\infty}\left[\frac{\partial y_{t+j}}{\partial w_t} + \frac{\partial y_{t+j}}{\partial w_{t+1}} + \ldots + \frac{\partial y_{t+j}}{\partial w_{t+j}}\right] = 1 + \phi + \phi^2 + \ldots = \frac{1}{1-\phi} \tag{12}$$

Finally we might wonder about the *cumulative consequences* for y of a one-time change in w. As in the former case we consider a transitory disturbance to w, but wish to calculate the sum of the consequences for all future values of y. It's evident that this is exactly the same as the long-run effect of a permanent change in w, i.e. equation (12).

4 The Software

We work in Excel. You already know all the required tools.

5 References

Chiang, Alpha C. (1984). *Fundamental Methods of Mathematical Economics*. McGraw-Hill.

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6 Todays Task

Exercise 1: the evolution of a dynamic system and the impulse-response functions

a) Let's consider equation (7). Suppose that $\phi = 0.7$. w_t is given by a constant term, let's say 0. Open a new spreadsheet in Excel. In a first column generate time $t = 0, 1, 2, \dots 20$. Supposing $y_{-1} = 1.5$ you can compute the evolution of y over time (the *time path* of y) in a separate column. Plot y against t (with time on the horizontal axis).

b) What are the effects of a change in ϕ on the time path of y (given that $w_t = 0$ for all t and y_{-1} remain unchanged)? When is the system *dynamically stable*?

c) Suppose that w_0 is increased by 1 with y_{-1} and w_1, w_2 , w_3, … taken as unaffected (a *temporary* change in w). Compute and plot the new time path for y.

d) Compute and plot the immediate and future effect on y of a temporary change in w, beginning in period 0 and ending in period 20. (Hint: The difference between the two time series computed in a) and c) gives you the immediate and future effects on y of a one-time change in w.) Is there another way to compute these effects?

e) Assume that w_t is permanently risen by 1, i.e. $w_t = 1$ for all t. Plot a time path for y.

f) What are the immediate and future effects on y of a permanent change in w, beginning in period 0 and ending in period 20?

g) What are the *cumulative consequences* of a temporary change in w? Compare with f).

Exercise 2: the cobweb model

a) Let's go back to equation (6). Given $a = 3.0$, $b = 0.51$, $c = 0.50$, $d = 0.60$, and initial price $P_0 = 3$. How does P_t evolve over time?

b) Assume different values for d and b. Show that there are three possible patterns. (Recall that $d, b > 0$.)

In order to illustrate the essence of the Hog Cycle Model imagine a conventional *Marshallian cross*. Given an initial price P_0, we can read off on the supply curve that the quantity supplied in the next period, period 1, will be Q_1 as a function of P_0 (according to the naive expectation formation described above). In order to clear the market, the quantity demanded in period 1 must also be Q_1, which is possible if and only if price is set at the level of P_1. Now, via the supply curve, the price P_1 lead to Q_2 as the quantity supplied in period 2, and to clear the market in the latter period, price must be set at the level of P_2 according to the demand curve. Repeating this reasoning we can trace out the prices and quantities in subsequent periods in the diagram, thereby spinning a *cobweb* around the demand and supply curves.

c) Try to plot a graph which shows the demand and supply curves and the cobweb. Suppose thereby that $P_0 = 2.5$.
Redo b) and watch the cobweb.