Session 12: The Solow Model

Doing Economics with the Computer

1 The Topic

Today's topic is the implementation of the Solow[1] model in Matlab. The Solow model is a growth model. Therefore, in contrast to e.g. the IS-LM model from Session 1 with which we explained short-term business cycle phenomena, it focuses on the behavior of economic variables in the long-run. Another difference to our simple version of the IS-LM model is that the Solow model is a dynamic model. The dynamics enter the model via capital accumulation. The distinguishing feature of the model is the assumption that investment equals a constant fraction (the so-called 'savings rate') of output.[2] The aim of this very basic formulation of a neoclassical model is to explain how the optimal level of output is determined and how this is allocated between consumption and capital accumulation, or to put it differently, how between consumption today and consumption in the future. Today's output can either be consumed or invested, and the existing capital stock can either be consumed today or used to produce output tomorrow.

[1]Named after Robert Merton Solow, Nobel laureate in economics in 1987.

[2]For a more comprehensive discussion of the Solow model see e.g. Mankiw (2009).

1.1 An Algebraic Version of the Solow Model

endogenous variables:

Y	output
C	consumption
I	investment
K	capital

exogenous variables:

K_1 initial capital stock

parameters:

A, N, α, s, δ

assumption 1: production function

$$Y_t = AF(K_t, N) = AK_t^{1-\alpha} N^{\alpha} \tag{1}$$

assumption 2: output equals aggregate demand

$$Y_t = C_t + I_t \tag{2}$$

assumption 3: investment as a constant fraction of output

$$I_t = sY_t \tag{3}$$

assumption 4: law of motion for the capital stock

$$K_{t+1} = (1-\delta)K_t + I_t \tag{4}$$

Equations (1), (3) and (4) can be combined to find the fundamental dynamic equation of the Solow model

$$K_{t+1} = sAK_t^{1-\alpha}N^{\alpha} + (1-\delta)K_t \tag{5}$$

1.1.1 Calibration

We assume the following values for the parameters and exogenous variable $A = 1$, $N = 4$, $\alpha = 2/3$, $s = 0.3$, $\delta = 0.1$, $K_1 = 5$.

2 Method and Software

The implementation of the Solow model in Matlab will allow us to work on several issues. Firstly, since it is a dynamic model it provides us with an opportunity to apply for-loops and while-loops. Furthermore, when considering the so-called *golden rule* we will encounter some optimization mechanisms of Matlab.

3 References

Mankiw, N. Gregory (2007). *Macroeconomics*, 6th edition, Worth Publishers.

4 Today's Task

First, let's consider the "steady state" solution of the model. The economy is at its long-run equilibrium. We can neglect the time index.

Exercise 1: steady state

a) Use the fundamental dynamic equation (5) and make a tentative back-of-the-envelope sketch with which you can find the so-called *steady states* (i.e. the points where the capital stock is constant over time: $K_{t+1} = K_t = K$)?

b) Open a new m-file and use equation (5) to compute the capital stock for $t = 2, \ldots, 100$ with a for-loop, thereby using the given parameter values and initial capital stock K_1. Plot the capital stock against time.

c) Using a for-loop, compute the steady state value of the capital stock. To do so, run the for-loop until two successive values of the capital stock are within a distance of, say, 0.0001. Then break the loop.

d) Compute the steady state values of the capital stock using a while-loop. Use the same criterion for convergence as in the previous exercise.

e) The analytically calculated steady state value of capital is

$$K = \left(\frac{sAN^{\alpha}}{\delta}\right)^{1/\alpha} = \left(\frac{sA}{\delta}\right)^{1/\alpha} N$$

Write a function m-file that, when given the values of the parameters $\{\alpha, s, A, \delta, N\}$, computes the theoretical steady state value of capital as well as the steady state values of Y, I and C (use equations (1), (3) and (2)).

f) Compare the results of 1.c) and 1.d) to the theoretical steady state value of capital computed in 1.e). Why is there a difference?

Exercise 2: the golden rule - optimal consumption

Now consider the problem of finding the so-called *golden rule* savings rate, i.e. the savings rate at which steady state consumption is maximized.

In order to find this optimal savings rate, it is easiest to find the level of capital that maximizes steady state consumption and then use the steady state equation of 1.e) to find the savings rate consistent with the chosen steady state level of capital. So formally, the problem is to maximize

$$C = AK^{1-\alpha}N^{\alpha} - \delta K \tag{6}$$

with respect to K. The resulting optimal stock of capital, K^*, is then used to find the optimal savings rate:

$$K^* = \left(\frac{s^* A}{\delta}\right)^{1/\alpha} N \Leftrightarrow s^* = \frac{\delta}{A}\left(\frac{K^*}{N}\right)^{\alpha} \tag{7}$$

Of course, in a simple model like this one we could solve this problem analytically. The necessary first-order condition (FOC) yields

$$\begin{aligned} \frac{\mathrm{d}C}{\mathrm{d}K} &\overset{!}{=} 0 \\ \Rightarrow \quad (1-\alpha)AK^{-\alpha}N^{\alpha} - \delta &= 0 \end{aligned} \tag{8}$$

The optimal level of capital would thus be

$$K^* = \left(\frac{(1-\alpha)A}{\delta}\right)^{1/\alpha} N \tag{9}$$

However, in order to practice, we will use various numerical optimization routines in Matlab to solve this problem.

a) Write a function m-file that calculates steady state consumption according to equation (6) using capital as an input (note: include the parameter values in the function m-file and do not provide them as function inputs). Use this function m-file together with the `fminbnd` command to maximize[3] consumption with respect to capital (use $K \in [0; 100]$ for the lower and upper bounds). Then use the optimal level of capital in equation (7) to compute the optimal savings rate.

b) Solve the problem of 2.a) using `fminsearch` instead of `fminbnd`.

c) An alternative way to find the level of capital that maximizes steady state consumption is to find the K that makes the FOC of the maximization problem hold, i.e. that sets the left-hand-side (LHS) of equation (8) equal to zero. In order to do so, write a new function m-file that calculates this LHS using capital as an input (note: include the parameter values in the function m-file and do not provide them as function inputs). Use this function m-file together with the `fcsolve` command to find the K that sets the LHS equal to zero. Try also the alternative command `fzero`. What do you find with regard to the starting values?

d) Compute the optimal level of steady state capital and the golden rule savings rate using the analytical formulas (i.e. equations (9) and (7)). Compare the results of the numerical and the theoretical optimization.

[3] To maximize C just minimize $-C$.