## Session 11: Financial Valuation

Harris Dellas*

## 1 Problem specification

In this lecture we will explain how to use MATLAB to solve two practical problems: a) amortization; b) interest compound

### 1.1 Mortgage Payments

A. Let us consider a fixed rate mortgage, that is, a mortgage that carries a fixed interest rate over its lifetime. We would like to understand how mortgage payments relate to the initial amount and the evolution of the principal, and the role of the interest rate and the length of the loan period. Let us assume that payments are made monthly until the principal has been paid out and that all the payments are of the same size.

## Definitions:

$T=$ maturity of loans (number of years)
$N=T X 12=$ total number of payments, $n=$ is number of payment
$p(t)=$ principal, $t=1,2, \ldots, N, A=$ initial principal
$R=$ constant, monthly payment
$r a=$ annual interest rate
An example: a 20 -year loan of size $A=350^{\prime} 000$, at an annual interest of $r a=0.05$, that is $5 \%$.
The principle then evolves according to

$$
\begin{equation*}
p(n+1)=(1+r) p(n)-R \tag{1}
\end{equation*}
$$

Noting the existence of a geometric progression in the representation above allows us to write the principal at the final date as

$$
\begin{equation*}
p(N)=A *(1+r)^{N}-R * \sum_{j=0}^{N-1}(1+r)^{j} \tag{2}
\end{equation*}
$$

The principal on the final date must be zero (the loan will have been fully repaid), that is $p(N)=0$. Using this in equation 2 and solving for R (utilizing the definition of the sum of a finite geometric series) gives the required amount of monthly payment $R$

$$
R=\frac{r * A *(1+r)^{N}}{(1+r)^{N}-1}
$$

[^0]Table 1: No of payments and remaining principal

| No of payments | Remaining Principal | Remaining Principal |
| :---: | :---: | :--- |
| 0 | $A=$ | A |
| 1 | $A *(1+r)-R=$ | $-R+A *(1+r)$ |
| 2 | $(A *(1+r)-R) *(1+r)-R=$ | $-R-R *(1+r)+A *(1+r)^{2}$ |
| 3 | $((A *(1+r)-R) *(1+r)-R) *(1+r)-R=$ | $-R-R *(1+r)-R *(1+r)^{2}+A *(1+r)^{3}$ |
| 4 | $(((A *(1+r)-R) *(1+r)-R) *(1+r)-R) *(1+r)-R=$ | $-R-R *(1+r)-R *(1+r)^{2}-R(1+r)^{3}+A *(1+r)^{4}$ |

## 2 Today's Tasks

### 2.1 Exercise 1

Suppose that $T=30, r a=0.05, A=400000$.
a) Using MATLAB determine the required monthly payments. How much principal is left after 10 years? After 20 years? Use a loop to answer the latter questions. Do it both with a script and using a function file.
b) Let the interest rate now be higher, namely 0.06 and answer part (a) again.
c) Suppose that given your income, you can afford to make a maximum monthly payment $R=500$. If $r=0.05$ what is the maximum amount of a 20 year loan that you can afford to borrow? Of a 30 year loan?
d) Suppose that you have an arrangement that during the first 5 years of the 30 year mortgage you only have to pay the interest due and nothing of the capital. What is the monthly payment you will have to make over
the remaining 25 years? Is it smaller or larger than if you did not have this deal? Compute the principal after 10 years. Plot the evolution of the principal with and without the deal in the same graph.
e) Suppose now that you pay nothing during the first 5 years and then you pay 500 over the following 5 years. What must the payment over the remaining 20 years be in order for the mortgage to be fully repaid?
f) Suppose that the bank unexpectedly forgives the remaining amount of the loan once the principal has fallen below 50000. After how many months will this happen?

### 2.2 Exercise 2

Let us now consider a flexible, 30 year mortgage, that is, a mortgage for which the interest rate changes from one month to the next. Let the monthly rate be $\mathrm{r}(\mathrm{n})$ and suppose, for simplicity, that the path of interest rates is known when the mortgage is issued and that the borrower still makes a fixed payment of 2000 every month. At the end of the 30 years period, the
borrower pays the balance to the creditor. How much is this balance?
Let $T=30, A=400000$ so that $N=360, R=2000$ and r be in the interval [a,b] of a uniform distribution, $a=0.003, b=0.005$, so that $r=$ $a+(b-a) * \operatorname{rand}(360,1)$.
a) Using MATLAB determine the amount of principal left after: 10 years; 30 years.
b) Now suppose that in each period you pay the full interest cost plus one tenth of the principal owed in that period. How much will you owe after 5 years? How many years will take you before your principle drops below 50000 ?

### 2.3 Interest rate compounding

Consider ${ }^{1}$ an account that has A dollars in it and pays a stochastic monthly interest r. Suppose that, beginning at a certain point, an amount $S$ is deposited monthly and no withdrawals are made. As in B above assume that $r=a+(b-a) . * \operatorname{rand}(240,1)$ with $a=0.003, b=0.005$.
a) Assume first that $S=0$. Compute the total T amount in the account after $n=60$ months. Assume that the interest is credited on the last day of the month and the total T is computed on the last day after the interest is credited.
b) Now assume that $\mathrm{A}=0$, that S is deposited on the first day of the month, and that as before interest is credited on the last day of the

[^1]month, and the total T is computed on the last day after the interest has been credited. Compute the total T in the account after $n=60$ months.
c) Combine the last two models assuming that there is an initial amount in the account (M) as well as a monthly deposit (S) and compute the total T in the account after $n=60$ months.

Now suppose that the interest rate is fixed through the deposit contract.
d) If the annual interest rate is $5 \%$, and no monthly deposits are made, how many years does it take to double your initial stash of money? What if the annual interest rate is $10 \%$ ?
e) In this and the next part, there is no initial stash. Assume an annual interest rate of $8 \%$. How much do you have to deposit monthly to be a millionaire in 35 years?
f) If the interest rate remains as in (e) and you can afford to deposit only 300 each month, how long do you have to work to retire a millionaire?
g) You hit the lottery and win 100,000 . You have two choices: take the money, pay the taxes, and invest what's left; or receive 100,000/240 monthly for 20 years, depositing what's left after taxes. Assume that a 100,000 windfall costs you 35,000 in federal and state taxes, but that the smaller monthly payoff causes only a $20 \%$ tax liability. In which way are you better off 20 years later? Assume a $5 \%$ annual interest rate here.
h) Historically, banks have paid roughly $5 \%$, while the stock market has tended to return $8 \%$ on average over a 10 -year period. So parts (e)
and (f) relate more to investing than to saving. But suppose that the market in a 5 -year period returns $13 \%, 15 \%,-3 \%, 5 \%$ and $10 \%$ in five successive years, and then repeats the cycle. (Note that the [arithmetic] average is $8 \%$, though a geometric mean would be more relevant here.) Assume that 50,000 is invested at the start of a 5 -year market period. How much does it grow to in 5 years? Now recompute four more times, assuming that you enter the cycle at the beginning of the second year, the third year, etc. Which choice yields the best/worst results? Can you explain why? Compare the results with a fixed-rate account paying $8 \%$. Assume simple annual interest. Redo the five investment computations, assuming that 10,000 is invested at the start of each year. Again analyze the results


[^0]:    *Slightly rearranged by Philipp Wegmueller

[^1]:    ${ }^{1}$ Borrowed (and modified) from "A Guide to MATLAB for Beginners and Experienced Users," Brian R. Hunt, Ronald L. Lipsman, Jonathan M. Rosenberg.

