Monetary policy in open economies: Technical Appendix

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1 Introduction

In this appendix I present a more general version of the model used in the paper "Monetary policy in open economies." The main differences are the following: a) The setting of nominal wages and prices involves adjustment costs. This mechanism has three advantages. It allows for some price adjustment within the period. It has interesting dynamics. And unlike the Calvo-Taylor specification it does not contain a distribution of firm prices. It is not unusual in second order approximations that the distribution term may exert undue numerical influence on the results. b) Goods and labor markets are imperfectly competitive in order to make the discussion of price adjustment for individual firms meaningful. c). The solution of the model is computed in a second order approximation. d) The welfare evaluations of alternative monetary regimes are performed using a second order approximation to the utility function, where the moments involved have been computed using the second order approximation to the model. e) The rate of physical capital depreciation falls short of unity. f) There is also a shock to preferences.

2 The model

The model consists of two economies. In each economy there are two types of firms. The first produces final goods and the second intermediate goods. We describe the behavior of the domestic economy. The behavior of the foreign economy is completely analogous.

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2.1 The final sector firms

Following standard practice in the literature we assume that the domestic final good y_h is produced by perfectly competitive domestic firms by combining domestic (x_{hh}) and imported (x_{hf}) intermediate goods. The final good y_h can be used for domestic private consumption and investment purposes. Its production is described by the following CES function

$$y_{ht} = \left(\omega^{1-\rho} x_{hht}^{\rho} + (1-\omega)^{1-\rho} x_{hft}^{\rho}\right)^{\frac{1}{\rho}} \tag{1}$$

where $\omega \in [0,1]$ and $\rho \in [-\infty, 1]$.

Minimization of total expenditures, $P_{hht}x_{hht} + P_{hft}x_{hft}$, where P_{hht} and P_{hft} denote the price of the domestic and the foreign bundle of goods in domestic currency, results in the standard demand equations:

$$x_{hht} = \left(\frac{P_{hht}}{P_{ht}}\right)^{\frac{1}{\rho-1}} \omega y_{ht} \quad \text{and} \quad x_{hft} = \left(\frac{P_{hft}^{\star} s_t}{P_{ht}}\right)^{\frac{1}{\rho-1}} (1-\omega) y_{ht}. \tag{2}$$

where P_{ht} is the general –CPI– price index at home. I assume producer currency pricing and purchasing power parity for traded goods. That is, $P_{hft} = s_t P_{hft}^{\star}$ where s_t is the nominal exchange rate and a $\star$ denotes foreign currency price.

$$P_{ht} = \left(\omega P_{hht}^{\frac{\rho}{\rho-1}} + (1-\omega)(s_t P_{hft}^{\star})^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}. \tag{3}$$

x_{hht} and x_{hft} are themselves combinations of the domestic and foreign intermediate goods, produced by each intermediate firm i, according to

$$x_t^{hh} = \left(\int_0^1 x_t^{hh}(i)^{\theta}\right)^{\frac{1}{\theta}} \; and \quad x_t^{fh} = \left(\int_0^1 x_{hft}(i)^{\theta}\right)^{\frac{1}{\theta}} \tag{4}$$

where $\theta \in [-\infty, 1]$.

2.2 The intermediate goods firms

Each intermediate firm $i \in [0,1]$ produces an intermediate good $x(i)$ using physical capital $k(i)$ and labor $h(i)$ according to a constant return-to-scale technology ($\alpha_k, \alpha_h \in [0,1]$, $\alpha_k + \alpha_h = 1$) represented by the production function

$$x_t(i) = \mathcal{A}_t k_t(i)^{\alpha_k} h_t(i)^{\alpha_h} \tag{5}$$

where $\mathcal{A}_t$ is an exogenous stationary stochastic technological shock.

Minimizing total labor expenditures, $\int_0^1 W_t(j)h_t(i,j)\mathrm{d}j$, leads to the following demand for labor of type j by firm i

$$h_t(i,j) = \left(\frac{W_t(j)}{W_t}\right)^{\frac{1}{\vartheta-1}} h_t(i) \tag{6}$$

where the aggregate nominal wage level is given by

$$W_t = \left(\int_0^1 W_t(j)^{\frac{\vartheta}{\vartheta-1}}\mathrm{d}j\right)^{\frac{\vartheta-1}{\vartheta}} \tag{7}$$

and $h(i)$ takes the form

$$h_t(i) = \left(\int_0^1 h_t(i,j)^{\vartheta}\mathrm{d}j\right)^{\frac{1}{\vartheta}}. \tag{8}$$

Assuming that each firm i operates under perfect competition in the input markets, it determines its production plan by minimizing total cost, $W_t h_t(i) + P_t z_t k_t(i)$, where z_t is the real rental of capital, subject to the production function (5). The input demand functions are given by (dropping i)

$$\alpha_k \psi_t P_t x_t = P_t z_t k_t \tag{9}$$

$$\alpha_h \psi_t P_t x_t = W_t h_t \tag{10}$$

where the real marginal cost, ψ_t is given by $\psi_t = \frac{z_t^{\alpha_k}(W_t/P_t)^{\alpha_h}}{\mathcal{A}_t\varsigma}$.

Intermediate goods producers are monopolistically competitive. Therefore, they set prices for the good they produce. It is assumed that they face an adjustment cost when they change their prices relative to some benchmark rate of inflation. Their profit maximization problem is given by

$$\max_{P_{xt}(i)} \left\{\mathbb{E}_t \sum_{n=0}^{\infty} D_{t,t+n}\Pi_{xt+n}(i)\right\} \tag{11}$$

where the discount factor is $D_{t,t} = 1$ and $D_{t,t+1} = \beta\frac{\Lambda_{t+1}(j)}{\Lambda_t(j)}$ and comes from the optimization problem of the household.

Profits are $\Pi_{xt}(i) = (P_{xt}(i) - P_t\psi_t)x_t(i) - \frac{\xi_x}{2}\left(\frac{P_{xt}(i)}{P_{xt-1}(i)} - \pi_x\right)^2 P_t y_t$. The last element represents the cost of changing prices relative to some benchmark rate of inflation, expressed in units of the final good. The first-order condition with regard to the choice of price, $P_{xt}(i)$, is

$$\frac{\theta}{\theta-1}x_t(i) - \frac{P_t\psi_t}{\theta-1}\frac{1}{P_{xt}(i)}x_t(i) - \frac{1}{P_{xt-1}(i)}\xi_x\left(\frac{P_{xt}(i)}{P_{xt-1}(i)} - \pi_x\right)P_t y_t$$
$$+\mathbb{E}_t D_{t,t+1}\frac{P_{xt+1}(i)}{P_{xt}(i)^2}\xi_x\left(\frac{P_{xt+1}(i)}{P_{xt}(i)} - \pi_x\right)P_{t+1}y_{t+1} = 0. \tag{12}$$

2.3 The Household

There exists a continuum of households, indexed by $j \in [0,1]$. The preferences of household j are given by

$$\mathbb{E}_t \sum_{\tau=0}^{\infty} \beta^{\tau} \left[\frac{\nu^{ct}}{1-\sigma_c} c_{t+\tau}(j)^{1-\sigma_c} - \frac{\nu^h}{1+\sigma_h} h_{t+\tau}(j)^{1+\sigma_h} \right] \tag{13}$$

where $0 < \beta < 1$ is a constant discount factor, $c_t(j)$ denotes the domestic consumption bundle, and $h_t(j)$ is the quantity of hours supplied by household of type j. ν^h is a constant while ν^{ct} represents a preference shock whose properties will be described below.

In each period, the representative household j faces a budget constraint

$$\begin{aligned} & E_t Q_t B_{t+1}(j) + M_t(j) + P_{ht} \left(c_t(j)(1+\eta_t(j)) + i_t(j) + \frac{\xi_w}{2} (\frac{W_t(j)}{W_{t-1}(j)} - \pi_w)^2 y_t \right) \\ = \;& B_t(j) + M_{t-1}(j) + P_t z_t k_t(j) + W_t(j) h_t(j) + N_t(j) + \Pi_t(j) \end{aligned} \tag{14}$$

where $B_t(j)$ are contingent claims in units of domestic currency with a price of Q_t, P_{ht}, is the general price index, $W_t(j)$ is the nominal wage, $i_t(j)$ is investment and $k_t(j)$ is the amount of capital owned by the household and leased to the firms at the real rental rate z_t. $M_{t-1}(j)$ is the amount of money that the household brings into period t, $M_t(j)$ is the end of period t money, and $N_t(j)$ is a nominal lump-sum transfer received from the monetary authority. $\Pi_t(j)$ denotes the profits distributed to the household by the firms. The expression $\frac{\xi_w}{2}\left(\frac{W_t(j)}{W_{t-1}(j)} - \pi_w\right)^2 P_t y_t$ captures the cost of adjusting nominal wages in terms of final good consumption. $\eta(v_t(j), \zeta_t)$ is a proportional monetary transaction cost that depends on ζ_t, a constant.

$$v_t(j) = \frac{P_t c_t(j)}{M_t(j)}. \tag{15}$$

The function η is borrowed from Schmitt-Grohe and Uribe, 2004.

$$\eta(v_t(j), \zeta) = \zeta_t \left(A v_t(j) + \frac{B}{v_t(j)} - 2\sqrt{AB} \right). \tag{16}$$

Capital accumulates according to

$$k_{t+1}(j) = i_t(j) - \frac{\varphi}{2} \left(\frac{i_t(j)}{k_t(j)} - \kappa \right)^2 k_t(j) + (1-\delta) k_t(j) \tag{17}$$

where $\delta \in [0,1]$ denotes the rate of depreciation. $\kappa > 0$ is a constant. The capital adjustment costs are assumed to be zero in the steady state.

The household then determines consumption/saving and money holdings maximizing (13) subject to (14) and (17).

The workers have monopoly power when selling their labor services. The first-order condition with regard to the choice of the nominal wage rate, $W_t(j)$, is obtained by maximizing (13) subject to (14) and the total demand for type j labor $h_t(j) = \int_0^1 h_t(i,j)$ and is given by

$$\frac{\vartheta}{\vartheta-1}h_t(j) - \frac{\frac{\nu^h h_t(j)^{1+\sigma_h}}{\Lambda_t(j)}}{\vartheta-1}\frac{1}{W_t(j)} - \frac{1}{W_{t-1}(j)}\xi_w\left(\frac{W_t(j)}{W_{t-1}(j)} - \pi_w\right)P_t y_t$$
$$+\mathbb{E}_t D_{t,t+1}\frac{W_{t+1}(j)}{W_t(j)^2}\xi_w\left(\frac{W_{t+1}(j)}{W_t(j)} - \pi_w\right)P_{t+1}y_{t+1} = 0. \qquad (18)$$

2.4 Monetary policy

We study two international and two domestic monetary arrangements: A flexible system and a bilateral peg; monetary targeting and a standard interest rate (Henderson-McKibbin and Taylor, HMT) rule. Under a flexible regime, the domestic rules take the form:

a) Strict monetary targeting (MT)

$$\frac{M_t - M_{t-1}}{M_{t-1}} = constant, \qquad (19)$$

b) A standard HMT rule

$$R_t = k_\pi(\pi_t - \pi) + k_y(y_t - y), \qquad (20)$$

where M_t is the money supply, R_t is the nominal interest rate, π_t is the inflation rate, π is the inflation target (equal to the steady state rate of inflation), y_t is output and y is the output target (equal to the steady state value of output). The last assumption is made in order to be consistent with the arguments about informational limitations in policy made in the paper.

Under a fixed regime, we assume that the nominal exchange rate is stabilized by changing the growth rates of the two money supplies in such a way that the rate of world money growth remains constant (global monetarism). Under the interest rate rules, we have each country's HTM rule also include a response to the inflation and output developments in the other country, so that the nominal interest rates in two countries remain always equal.

3 Calibration

We are mostly interested in establishing results that hold for a "generic" rather than for a particular, real world economy. Consequently, we rely mostly on parameters that are commonly used in the open economy literature. The benchmark parameters are reported in table 1.

Table 1: Calibration, benchmark case

Parameter		Value
Production	α_k	0.2268
Wage markup	ϑ	0.8000
Invest. adjust. cost	φ	10.0000
Depreciation	δ	0.0250
Trade elasticity	ρ	0.8000
Markup goods	θ	0.8000
Trade share	ω	0.8500
Discount factor	β	0.9900
Utility	σ_h	1.0000
Utility	σ_c	1.5000
Preferences	ν^c	1.0000
Preferences	ν^h	1.0000
Inflation rate	π	1.0096

There is not much information in the literature regarding the appropriate range of values for the parameters ξ of nominal prices and wage adjustment costs. We use a value of 20, which seems rather modest. For instance, Ireland, 2000, suggests a value of 50.

All shocks are assumed to follow independent AR(1) processes with an autoregressive coefficient of 0.9. In the experiments run, the standard deviation of the shocks have been set to 0.01.

4 The results

Tables 1-7 present welfare comparisons (the welfare computations are based on Collard and Dellas, 2005, and are described in greater detail further below) for the different monetary arrangements. The main patterns are as follows: First, rigid rules (money supply or exchange rate targeting) do better under nominal wage rigidity, more or less independent of the type of the shock and the specification of the interest rate policy rule. Second, in the benchmark specification of policy, the differences across exchange rate regimes tend to be small relative to the differences across domestic, operating procedures. And third, under nominal price rigidity, the comparison of the alternative regimes hinges on the degree of price rigidity and the specification of the interest rate rule. The greater the degree of nominal rigidity, the greater (smaller) the weight in the rule on inflation (output), and the use of the GDP deflator instead of the CPI , the more likely that a flexible exchange rate system with an interest rate policy rule will outperform the other monetary arrangements.

Table 1: Welfare comparisons: Wage rigidity, $\xi_w = 20$, $\xi_p = 0$, $k_\pi = 1.5$

shock	FLEX-M	FLEX-R	FIX-M	FIX-R
Supply	-388.2264	-388.7000	-388.2330	-388.3238
Demand	-390.3228	-390.4638	-390.2900	-390.2876
Fiscal	-388.2291	-388.2947	-388.2287	-388.2294

Note: Each entry gives the level of welfare for each shock.

Table 2: Welfare comparisons: Price rigidity, $\xi_w = 0$, $\xi_p = 20$,$k_\pi = 1.5$

shock	FLEX-M	FLEX-R	FIX-M	FIX-R
Supply	-388.3521	-388.7542	-388.3623	-388.4178
Demand	-390.3176	-390.4377	-390.2884	-390.2863
Fiscal	-388.2289	-388.2821	-388.2285	-388.2288

Note: Each entry gives the level of welfare for each shock.

Table 3: Welfare comparisons: Wage rigidity, $\xi_w = 20$, $\xi_p = 0$, $k_\pi = 100$

shock	FLEX-M	FLEX-R	FIX-M	FIX-R
Supply	-388.2264	-388.2380	-388.2330	-388.2667
Demand	-390.3228	-390.2933	-390.2900	-390.2881
Fiscal	-388.2291	-388.2322	-388.2287	-388.2297

Note: Each entry gives the level of welfare for each shock.

Table 4: Welfare comparisons: Price rigidity, $\xi_w = 0$, $\xi_p = 20$, $k_\pi = 100$

shock	FLEX-M	FLEX-R	FIX-M	FIX-R
Supply	-388.3521	-388.2921	-388.3623	-388.3273
Demand	-390.3176	-390.2900	-390.2884	-390.2864
Fiscal	-388.2289	-388.2307	-388.2285	-388.2289

Note: Each entry gives the level of welfare for each shock.

Table 5: Welfare comparisons: Price rigidity, GDP deflator targeting, $\xi_w = 0$, $\xi_p = 20$, $k_\pi = 1.5$,$k_y = 0$

shock	FLEX-M	FLEX-R	FIX-M	FIX-R
Supply	-388.3521	-388.3125	-388.3623	-388.3287

Note: Each entry gives the level of welfare for each shock.

Table 6: Macroeconomic volatility: All shocks, price rigidity

	FLEX-M	FLEX-R	FIX-M	FIX-R
c	0.017	0.019	0.017	0.019
h	0.008	0.005	0.008	0.006
p	0.015	0.098	0.017	0.026
px	0.017	0.100	0.019	0.027
y	0.016	0.017	0.017	0.018

Note: Standard deviation of output, y, employment, h, CPI, p, domestic price level, p_h and consumption, c.

Table 7: Macroeconomic volatility: All shocks, wage rigidity

	FLEX-M	FLEX-R	FIX-M	FIX-R
c	0.019	0.020	0.019	0.022
h	0.007	0.009	0.006	0.012
p	0.017	0.109	0.019	0.052
px	0.020	0.111	0.021	0.053
y	0.019	0.019	0.019	0.023

Note: Standard deviation of output, y, employment, h, CPI, p, domestic price level, p_h and consumption, c.

References

Collard, F. and H. Dellas, 2005, Tax Distortions and the Case for Price Stability, *Journal of Monetary Economics*, January.

Ireland, P., 2000, Interest Rates, Inflation, and Federal Reserve Policy since 1980, *Journal of Money Credit and-Banking*, Part 1, 32(3): 417-34.

Schmitt-Grohé, S. and M. Uribe, 2004, Optimal Fiscal and Monetary Policy Under Sticky Prices, *Journal of Economic Theory*, 114(2), 198-230.

4.1 Welfare Calculation

Utility can be approximated around the efficient economy by

$$u(c_t, 1-h_t) \simeq \overline{u}+\overline{u}_c(c_t-\overline{c})-\overline{u}_\ell(h_t-\overline{h})+\frac{1}{2}\overline{u}_{cc}(c_t-\overline{c})^2+\frac{1}{2}\overline{u}_{\ell\ell}(h_t-\overline{h})^2-\overline{u}_{c\ell}(c_t-\overline{c})(h_t-\overline{h})$$

Further, we can approximate c_t and h_t around the deterministic steady state by

$$c_t \simeq c^\star\left(1+\widehat{c}_t+\frac{\widehat{c}_t}{2}\right) \text{ and } h_t \simeq h^\star\left(1+\widehat{h}_t+\frac{\widehat{h}_t}{2}\right)$$

Therefore

$$c_t-\overline{c} \simeq (c^\star-\overline{c})+c^\star\left(\widehat{c}_t+\frac{\widehat{c}_t^2}{2}\right) \text{ and } h_t-\overline{h} \simeq (h^\star-\overline{h})+h^\star\left(\widehat{h}_t+\frac{\widehat{h}_t^2}{2}\right)$$

We have an approximation of order 2 of the form

$$\begin{aligned}
\widehat{c}_t &\simeq \pi_c\widehat{x}_t+\frac{1}{2}\widehat{x}_t'H_c\widehat{x}_t+\frac{\varepsilon_c}{2}\\
\widehat{h}_t &\simeq \pi_h\widehat{x}_t+\frac{1}{2}\widehat{x}_t'H_h\widehat{x}_t+\frac{\varepsilon_h}{2}
\end{aligned}$$

where $\widehat{x}_t$ is the vector of state variables. We then have

$$\begin{aligned}
E(\widehat{c}_t) &= \frac{\eta_c^2}{2}+\frac{\varepsilon_c}{2}\\
E(\widehat{c}_t^2) &= \sigma_c^2+\frac{\varepsilon_c\eta_c^2}{2}+\frac{\varepsilon_c^2}{4}\\
E(\widehat{h}_t) &= \frac{\eta_h^2}{2}+\frac{\varepsilon_h}{2}\\
E(\widehat{h}_t^2) &= \sigma_h^2+\frac{\varepsilon_h\eta_h^2}{2}+\frac{\varepsilon_h^2}{4}\\
E(\widehat{c}_t\widehat{h}_t) &= \sigma_{ch}+\frac{\varepsilon_h\eta_c^2}{4}+\frac{\varepsilon_c\eta_h^2}{4}+\frac{\varepsilon_c\varepsilon_h}{4}
\end{aligned}$$

where $\eta_c^2 = H_c E(\widehat{x}_t^2)$ and $\eta_h^2 = H_h E(\widehat{x}_t^2)$ with $E(\widehat{x}_t^2) \equiv \text{vec}E(\widehat{x}_t'\widehat{x}_t)$. We also have $\sigma_c^2 = \pi_c E(\widehat{x}_t\widehat{x}_t')\pi_c'$, $\sigma_h^2 = \pi_h E(\widehat{x}_t\widehat{x}_t')\pi_h'$ and $\sigma_{ch} = \pi_c E(\widehat{x}_t\widehat{x}_t')\pi_h'$. Note that here we neglected terms of order higher than 2. Then, we have

$$
\begin{aligned}
E(c_t-\overline{c}) &= (c^\star-\overline{c})+c^\star\left(E\widehat{c}_t+\frac{E\widehat{c}_t^2}{2}\right)\\
E(c_t-\overline{c})^2 &= (c^\star-\overline{c})^2+c^{\star 2}E\widehat{c}_t^2+2(c^\star-\overline{c})c^\star E\widehat{c}_t+c^\star(c^\star-\overline{c})E\widehat{c}_t^2\\
E(h_t-\overline{h}) &= (h^\star-\overline{h})+h^\star\left(E\widehat{h}_t+\frac{E\widehat{h}_t^2}{2}\right)\\
E(h_t-\overline{h})^2 &= (h^\star-\overline{h})^2+h^{\star 2}E\widehat{h}_t^2+2(h^\star-\overline{h})h^\star E\widehat{h}_t+h^\star(h^\star-\overline{h})E\widehat{h}_t^2\\
E(c_t-\overline{c})(h_t-\overline{h}) &= (c^\star-\overline{c})(h^\star-\overline{h})+h^\star(c^\star-\overline{c})E\widehat{h}_t+c^\star(h^\star-\overline{h})E\widehat{c}_t+c^\star h^\star E\widehat{c}_t\widehat{h}_t\\
&\quad +h^\star(c^\star-\overline{c})\frac{E\widehat{h}_t^2}{2}+c^\star(h^\star-\overline{h})\frac{E\widehat{c}_t^2}{2}
\end{aligned}
$$

once again neglecting terms of order higher than 2.

These terms all involve computation of expectations which we now detail. The solution of the model takes the form

$$X_{t+1} = M_{\text{X}}X_t+\frac{H_{\text{X}}}{2}X_t^2+M_{\text{E}}\varepsilon_{t+1}+\frac{\eta_{\text{X}}}{2} \tag{21}$$

$$Y_t = M_{\text{Y}}X_t+\frac{H_{\text{Y}}}{2}X_t^2+\frac{\eta_{\text{Y}}}{2} \tag{22}$$

where $X_t^2 = \text{vec}(X_tX_t')$. We want to compute first and second order moments for this solution. We first deal with the state equation.

$$E(X_{t+1}) = M_{\text{X}}E(X_t)+\frac{H_{\text{X}}}{2}E(X_t^2)+\frac{\eta_{\text{X}}}{2} \tag{23}$$

implying that

$$E(X_t) = K\left(\frac{\eta_{\text{X}}}{2}+\frac{H_{\text{X}}}{2}E(X_t^2)\right)$$

where $K = (I-M_{\text{X}})^{-1}$.

$$
\begin{aligned}
E(X_{t+1}X_{t+1}') =& M_{\text{X}}E(X_tX_t')M_{\text{X}}'+M_{\text{X}}E(X_tX_t^{2'})\frac{H_{\text{X}}'}{2}+M_{\text{X}}E(X_t\varepsilon_{t+1}')M_{\text{E}}'+M_{\text{X}}E(X_t)\frac{\eta_{\text{X}}'}{2}\\
&+\frac{H_{\text{X}}}{2}E(X_t^2X_t')M_{\text{X}}'+\frac{H_{\text{X}}}{2}E(X_t^2X_t^{2'})\frac{H_{\text{X}}'}{2}+\frac{H_{\text{X}}}{2}E(X_t^2\varepsilon_{t+1}')M_{\text{E}}'+\frac{H_{\text{X}}}{2}E(X_t^2)\frac{\eta_{\text{X}}'}{2}\\
&+M_{\text{E}}E(\varepsilon_{t+1}X_t')M_{\text{X}}'+M_{\text{E}}E(\varepsilon_{t+1}X_t^{2'})\frac{H_{\text{X}}'}{2}+M_{\text{E}}E(\varepsilon_{t+1}\varepsilon_{t+1}')M_{\text{E}}'+M_{\text{E}}E(\varepsilon_{t+1})\frac{\eta_{\text{X}}'}{2}\\
&+\frac{\eta_{\text{X}}}{2}E(X_t')M_{\text{X}}'+\frac{\eta_{\text{X}}}{2}E(X_t^{2'})\frac{H_{\text{X}}'}{2}+\frac{\eta_{\text{X}}}{2}E(\varepsilon_{t+1}')M_{\text{E}}'+\frac{\eta_{\text{X}}}{2}\frac{\eta_{\text{X}}'}{2}
\end{aligned} \tag{24}
$$

Neglecting terms of order greater than 2, acknowledging that $E(\varepsilon_{t+1}) = 0$ and $E(\varepsilon_{t+1}X_t') = 0$, equation (24) reduces to

$$\begin{aligned} E(X_{t+1}X_{t+1}') =& M_{\rm X}E(X_tX_t')M_{\rm X}' + M_{\rm X}E(X_t)\frac{\eta_{\rm X}'}{2} + \frac{H_{\rm X}}{2}E(X_t^2)\frac{\eta_{\rm X}'}{2} + P\Sigma P' \\ &+ \frac{\eta_{\rm X}}{2}E(X_t')M_{\rm X}' + \frac{\eta_{\rm X}}{2}E(X_t^{2'})\frac{H_{\rm X}'}{2} + \frac{\eta_{\rm X}\eta_{\rm X}'}{4} \end{aligned} \tag{25}$$

Using the solution for $E(X_t)$, we have

$$\begin{aligned} E(X_{t+1}X_{t+1}') =& M_{\rm X}E(X_tX_t')M_{\rm X}' + M_{\rm X}K\frac{\eta_{\rm X}\eta_{\rm X}'}{4} + M_{\rm X}K\frac{H_{\rm X}}{2}E(X_t^2)\frac{\eta_{\rm X}'}{2} + \frac{H_{\rm X}}{2}E(X_t^2)\frac{\eta_{\rm X}'}{2} + P\Sigma P' \\ &+ \frac{\eta_{\rm X}}{2}\frac{\eta_{\rm X}'}{2}K'M_{\rm X}' + \frac{\eta_{\rm X}}{2}E(X_t^{2'})\frac{H_{\rm X}'}{2}K'M_{\rm X}' + \frac{\eta_{\rm X}}{2}E(X_t^{2'})\frac{H_{\rm X}'}{2} + \frac{\eta_{\rm X}\eta_{\rm X}'}{4} \end{aligned} \tag{26}$$

Taking the vector form of the latter equation, we have

$$\text{vec}\Sigma_{\rm XX} = Q^{-1}\text{vec}\left(M_{\rm X}K\frac{\eta_{\rm X}\eta_{\rm X}'}{4} + P\Sigma P' + \frac{\eta_{\rm X}\eta_{\rm X}'}{4}K'M_{\rm X}' + \frac{\eta_{\rm X}\eta_{\rm X}'}{4}\right)$$

where $Q \equiv \left[I_{n_x^2} - M_{\rm X}\otimes M_{\rm X} - \frac{1}{4}(\eta_{\rm X}\otimes M_{\rm X}KH_{\rm X}) - \frac{1}{4}(\eta_{\rm X}\otimes H_{\rm X}) - \frac{1}{4}(H_{\rm X}KM_{\rm X}\otimes\eta_{\rm X}) - \frac{1}{4}(H_{\rm X}\otimes\eta_{\rm X})\right]$, and $\Sigma_{\rm XX} \equiv E(X_tX_t')$. We can then get

$$E(X_t) = K\left(\frac{\eta_{\rm X}}{2} + \frac{H_{\rm X}}{2}\text{vec}\Sigma_{\rm XX}\right)$$

From (22), we can compute $E(Y_t)$ and $E(Y_tY_t')$ as

$$E(Y_t) = M_{\rm Y}E(X_t) + \frac{H_{\rm Y}}{2}\text{vec}\Sigma_{\rm XX} + \frac{\eta_{\rm Y}}{2}$$

$$E(Y_tY_t') = M_{\rm Y}\Sigma_{\rm XX}M_{\rm Y}' + M_{\rm Y}E(X_t)\frac{\eta_{\rm Y}'}{2} + \frac{H_{\rm Y}}{2}\text{vec}\Sigma_{\rm XX}\frac{\eta_{\rm Y}'}{2} + \frac{\eta_{\rm Y}}{2}E(X_t')M_{\rm Y}' + \frac{\eta_{\rm Y}}{2}(\text{vec}\Sigma_{\rm XX})'\frac{H_{\rm Y}'}{2} + \frac{\eta_{\rm Y}\eta_{\rm Y}'}{4}$$