Wage Rigidity and Monetary Union
Technical Appendix

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Abstract

We compare monetary union to flexible exchange rates in an asymmetric, three-country model with active monetary policy. Unlike the traditional OCA literature, we find that countries with high nominal wage rigidities benefit from monetary union, specially when they join other, similarly rigid countries. Countries with relatively more flexible wages lose when they form a union with more rigid wage countries. We study the France, Germany and the UK and find that wage asymmetries across these three countries dominate other types of asymmetries (in shocks, monetary policy etc.) in welfare comparisons. And that, if the UK had a substantially higher degree of wage flexibility relative to France and Germany, then her participation in EMU would be costly.

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1 The model

The three countries are modelled in a similar fashion [1] so we describe only one country, let us say the UK (see the appendix for the rest).

The economy consists of a large number of identical households and firms, a fiscal authority and a monetary authority.

1.1 The Household

The household maximizes expected lifetime utility:

$$E_0[\sum_{t=\infty} \beta^t U(C_t^S, h_t^S)] \tag{1}$$

where $0 < \beta < 1$ is a constant discount factor, C_t^S denotes UK consumption in period t and h_t^S is the number of hours worked by the UK representative household. $U(C_t^S, h_t^S)$ is a utility function, increasing and concave in its first argument, and decreasing and convex in its last argument. The following utility function will be used:

$$U(C_t^S, h_t^S) = \log(C_t^S) + \theta \log(1 - h_t^S) \tag{2}$$

where θ is a weight for the marginal utility of leisure.

In each and every period the UK household faces the following budget constraint:

$$\begin{aligned} & P_t^S C_t^S + P_t^S I_t^S + \int_\ell (\frac{\widetilde{P}_t^F}{e_t^S} B_{S,t+1}^F + \frac{e_t^G}{e_t^S} \widetilde{P}_t^G B_{S,t+1}^G + \widetilde{P}_t^S B_{S,t+1}^S) d\ell + M_{t+1}^S + P_t^S T_t^S \\ = \quad & W_t^S h_t^S + z_t^S K_t^S + \Pi_t^S + \frac{B_{S,t}^F}{e_t^S} + \frac{e_t^G}{e_t^S} B_{S,t}^G + B_{S,t}^S + M_t^S + N_t^S \end{aligned} \tag{3}$$

where P_t^S denotes the price of UK consumption and investment goods, I_t^S is investment, e_t^S is the FF/SF exchange rate, e_t^G is the FF/DM rate (hence e_t^G/e_t^S

[1] Nevertheless they may still differ in terms of size, economic structure, shocks and so on.

is the SF/DM rate), $\widetilde{P}_t^j$ is the price paid for an asset that will deliver 1 unit of country j's currency (j = F, G, S) next period if state ℓ realizes. A typical UK household owns $B_{S,t}^j$ such assets entering period t. M_t^S is the stock of money held by the UK household in period t, T_t^S is *lump-sum taxes*, W_t^S is the nominal wage, z_t^S is the rental rate for capital, K_t^S is the physical capital stock at the beginning of period t, Π_t^S are the profits of the UK firms and N_t^S is a per-capita amount of money issued by the Bank of England (BoE) and given to the households in the form of a helicopter drop.

According to the budget constraint, the households enters period t holding an amount of money equal to M_t; it receives income from its financial investments, $B_{S,t}^j$, from its labor services, from renting capital to the firms. It also receives its share of the profits distributed by the firms and its share of the money injection by the BoE. It uses these funds to buy new financial assets, to build its cash reserves, to pay taxes and to purchase goods for consumption and investment purposes.

Physical capital accumulates according to

$$K_{t+1}^S = \Phi(\frac{I_t^S}{K_t^S})K_t^S + (1+\delta)K_t^S \tag{4}$$

where $0 \leq \delta \leq 1$ denotes the rate of depreciation. The concave function $\Phi(.)$ captures the presence of adjustment cost to investment. It is assumed to be twice differentiable and homogenous of degree 0. Furthermore, we assume the absence of adjustment cost in the steady state: $\Phi(\gamma+\delta-1) = \gamma+\delta-1$, $\Phi'(\gamma+\delta-1) = 1$ and $\frac{\Phi''(\gamma+\delta-1)(\gamma+\delta-1)}{\Phi'(\gamma+\delta-1)} = \varphi$.

In each and every period t, the household faces a cash-in-advance (CIA) constraint on consumption purchases:

$$P_t^S C_t^S \leq M_t^S \tag{5}$$

Finally, we will assume that –at least a fraction of– the nominal wages are fixed one period in advance at a level that is equal to the expected labor market clearing wage, $\widetilde{W}_t$:

$$W_t = E_{t-1}\widetilde{W}_t \tag{6}$$

The households that have signed labor contracts must then supply whatever quantity of labor is demanded by the firms.

1.2 The firms

There are two types of firms, those that produce an intermediate good, Y, and those that produce a final good, Q.

The production of the intermediate good is done according to:

$$Y_t^S = a_t^S (K_t^S)^\alpha (\Gamma_t h_t^S)^{1-\alpha} \tag{7}$$

where K_t denotes the physical capital stock at the beginning of period t. Γ_t represents Harrod neutral, deterministic, technical progress evolving according to $\Gamma_t = \gamma \Gamma_{t-1}$. $\gamma \geq 1$ denotes the deterministic rate of growth. a_t^S is a stationary, exogenous, stochastic technological shock.[2]

The representative intermediate good firm chooses the quantity of capital and labor to lease in period t in order to maximize its current profits

$$\pi_t = (P_{Yt}^S Y_t^S - W_t^S h_t^S - z_t^S K_t^S) \tag{8}$$

where P_{Yt}^S is the price of the UK intermediate good.

The country specific intermediate goods are then combined to produce the final goods in the three countries.

$$Y_t^S = Y_{Ft}^S + Y_{Gt}^S + Y_{St}^S \tag{9}$$

where $Y_{j,t}^S$ denotes the amount of UK intermediate good that is used as an input to produce country j's final good in period t.

1.3 Production of the final domestic good

The production of the final good in the UK, Q_t^S, takes place according to:

$$Q_t^S = [\varpi_4^{1-\rho}(Y_{S,t}^F)^\rho + \varpi_5^{1-\rho}(Y_{S,t}^G)^\rho + \varpi_6^{1-\rho}(Y_{S,t}^S)^\rho]^{\frac{1}{\rho}} \tag{10}$$

The level of production is selected in order to maximize profits:

[2] The stochastic properties of the technology shock will be described in section **??**.

$$\pi^S = P_t^S Q_t^S - \frac{P_{Yt}^F}{e_t^S} Y_{S,t}^F - \frac{e_t^G}{e_t^S} P_{Yt}^G Y_{S,t}^G - P_{Yt}^S Y_{S,t}^S \tag{11}$$

where ϖ_4, is the weight of the French goods in the UK final good basket, ϖ_5, is the weight of German goods in this basket and ϖ_6 denotes the weight of UK goods in the domestic (UK) basket. Recall that $Y_{F,t}^j$ is the amount of the intermediate good of country j (j = F, G, S) used in the production of the UK final good. $\frac{1}{\rho-1}$ is the elasticity of substitution between the domestic and foreign intermediate goods.

Clearing of the UK final good market requires:

$$Q_t^S = C_t^S + I_t^S + G_t^S \tag{12}$$

where G^S is UK government expenditure.

1.4 Government

In each period the government acquires an amount G_t of the final good. The cyclical component of government expenditures ($g_t = G_t/\Gamma_t$) is exogenously determined by a stationary AR(1) process such that:

$$\log(g_t) = \rho_g \log(g_{t-1}) + (1-\rho_g)\log(g) + \varepsilon_{gt} \tag{13}$$

with $|\rho_g| < 1$ and $\varepsilon_{gt} \rightsquigarrow \mathcal{N}(0, \sigma_g)$.

These expenditures are financed by means of lump–sum taxation

$$P_t^S G_t^S = P_t^S T_t^S \tag{14}$$

1.5 Monetary authorities

The behavior of the monetary authorities depends on the international monetary arrangement in place. Under a flexible exchange rate regime, we assume that monetary authorities pursue active monetary policy. In particular, central banks are assumed to follow a Taylor rule of the form

$$\widehat{R}_t^S = \rho^S \widehat{R}_{t-1}^S + (1-\rho^S)[\dot{K}_y^S E_t[\widehat{Y}_{t+1}^S] + \dot{K}_\Pi^S E_t[\widehat{\Pi}_{t+1}^S]] + \zeta_{r,t}^S \tag{15}$$

where R_t^S is the gross nominal interest rate, ρ^S denotes the degree of interest rate smoothing, $E_t[\widehat{Y}_{t+1}^S]$ is expected output (relative to target) and $E_t[\widehat{\Pi}_{t+1}^S]$ is expected CPI inflation (relative to target). K_y^S and K_Π^S are fixed weights.

In all cases we will assume that the supply of money involves according to

$$M_{t+1}^S = \mu_t^S M_t^S \tag{16}$$

where μ_t is the gross rate of growth. This is selected endogenously in order to satisfy the constraint imposed by the nominal interest rate policy. Under a unilateral French peg, France is assumed to select the growth rate of its supply of money, μ_t, in order to maintain a fixed FF/DM rate (while the Bundesbank pursues its Taylor rule). This policy is implemented by solving for the exchange rate as a function of the state variables of the system (a set that includes μ_t) and then selecting a value for μ_t that satisfies the exchange rate target, e.

2 The solution

This section describes in detail the computation of the equilibrium in this economy. We start by deriving the first order conditions that define the optimal behavior of each group of agents in these economies.

2.1 First order conditions

Households

French households

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left(\log(C_t^F) + \theta \log(1 - h_t^F)\right)$$

subject to

$$\int_\ell \left(p_{bt}^F B_{Ft+1}^F + e_t^G p_{bt}^G B_{Ft+1}^G + e_t^S p_{bt}^S B_{Ft+1}^S\right) d\ell + M_{t+1}^F + P_t^F C_t^F + P_t^F I_t^F + P_t^F T_t^F \leqslant$$

$$B_{Ft}^F + e_t^G B_{Ft}^G + e_t^S B_{Ft}^S + M_t^F + N_t^F + W_t^F h_t^F + z_t^F K_t^F$$

$$K_{t+1}^F = \Phi\left(\frac{I_t^F}{K_t^F}\right) K_t^F + (1-\delta) K_t^F$$

$P_t^F C_t^F \leqslant M_t^F$

Noting that the cash in advance constraint (CIA) will be satisfied as an equality if nominal interest rates are positive and substituting this constraint into the budget constraint produces the following first order conditions:

$$P_t^F \Lambda_{Mt}^F = \Phi'\left(\frac{I_t^F}{K_t^F}\right)\Lambda_{Kt}^F$$

$$\frac{\theta}{1-h_t^F} = \Lambda_{Mt}^F W_t^F$$

$$\Lambda_{Mt}^F = \frac{\beta}{M_{t+1}^F}$$

$$\Lambda_{Kt}^F = \beta E_t\left[\Lambda_{Mt+1}^F z_{t+1}^F + \Lambda_{Kt+1}^F\left(\Phi\left(\frac{I_{t+1}^F}{K_{t+1}^F}\right) - \frac{I_{t+1}^F}{K_{t+1}^F}\Phi'\left(\frac{I_{t+1}^F}{K_{t+1}^F}\right) + 1 - \delta\right)\right]$$

$$P_{bt}^F = \beta\frac{\Lambda_{Mt+1}^F}{\Lambda_{Mt}^F} f(s|s_t)$$

$$P_{bt}^G = \beta\frac{e_{t+1}^G}{e_t^G}\frac{\Lambda_{Mt+1}^F}{\Lambda_{Mt}^F} f(s|s_t)$$

$$P_{bt}^S = \beta\frac{e_{t+1}^S}{e_t^S}\frac{\Lambda_{Mt+1}^F}{\Lambda_{Mt}^F} f(s|s_t)$$

where Λ_{Kt}^F and Λ_{Mt}^F are the Langrange multipliers associated with the capital accumulation and the budget constraint respectively

German households:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left(\log(C_t^G) + \theta\log(1-h_t^G)\right)$$

subject to

$$\int_{\ell}\left(\frac{p^F_{bt}}{e^G_t}B^F_{Gt+1}+p^G_{bt}B^G_{Gt+1}+\frac{e^S_t}{e^G_t}p^S_{bt}B^S_{Gt+1}\right)d\ell+M^G_{t+1}+P^G_tC^G_t+P^G_tI^G_t+P^G_tT^G_t\leqslant$$

$$\frac{1}{e^G_t}B^F_{Gt}+B^G_{Gt}+\frac{e^S_t}{e^G_t}B^S_{Gt}+M^G_t+N^G_t+W^G_th^G_t+z^G_tK^G_t$$

$$K^G_{t+1}=\Phi\left(\frac{I^G_t}{K^G_t}\right)K^G_t+(1-\delta)K^G_t$$

$$P^G_tC^G_t\leqslant M^G_t$$

First order conditions:

$$P^G_t\Lambda^G_{Mt} = \Phi'\left(\frac{I^G_t}{K^G_t}\right)\Lambda^G_{Kt}$$

$$\frac{\theta}{1-h^G_t} = \Lambda^G_{Mt}W^G_t$$

$$\Lambda^G_{Mt} = \frac{\beta}{M^G_{t+1}}$$

$$\Lambda^G_{Kt} = \beta E_t\left[\Lambda^G_{Mt+1}z^G_{t+1}+\Lambda^G_{Kt+1}\left(\Phi\left(\frac{I^G_{t+1}}{K^G_{t+1}}\right)-\frac{I^G_{t+1}}{K^G_{t+1}}\Phi'\left(\frac{I^G_{t+1}}{K^G_{t+1}}\right)+1-\delta\right)\right]$$

$$P^F_{bt} = \beta\frac{e^G_t}{e^G_{t+1}}\frac{\Lambda^G_{Mt+1}}{\Lambda^G_{Mt}}f(s|s_t)$$

$$P^G_{bt} = \beta\frac{\Lambda^G_{Mt+1}}{\Lambda^G_{Mt}}f(s|s_t)$$

$$P^S_{bt} = \beta\frac{e^G_t}{e^G_{t+1}}\frac{e^S_{t+1}}{e^S_t}\frac{\Lambda^G_{Mt+1}}{\Lambda^G_{Mt}}f(s|s_t)$$

UK households:

$$\max E_0\sum_{t=0}^{\infty}\beta^t\left(\log(C^S_t)+\theta\log(1-h^S_t)\right)$$

subject to

$$\int_\ell \left(\frac{p^F_{bt}}{e^S_t} B^F_{St+1} + \frac{e^G_t}{e^S_t} p^G_{bt} B^G_{St+1} + p^S_{bt} B^S_{St+1} \right) d\ell + M^S_{t+1} + P^S_t C^S_t + P^S_t I^S_t + P^S_t T^S_t \leqslant$$

$$\frac{1}{e^S_t} B^F_{St} + \frac{e^G_t}{e^S_t} B^G_{St} + B^S_{St} + M^S_t + N^S_t + W^S_t h^S_t + z^S_t K^S_t + \Pi^S_t$$

$$K^S_{t+1} = \Phi \left(\frac{I^S_t}{K^S_t} \right) K^S_t + (1-\delta) K^S_t$$

$$P^S_t C^S_t \leqslant M^S_t$$

First order conditions:

$$P^S_t \Lambda^S_{Mt} = \Phi' \left(\frac{I^S_t}{K^S_t} \right) \Lambda^S_{Kt}$$

$$\frac{\theta}{1-h^S_t} = \Lambda^S_{Mt} W^S_t$$

$$\Lambda^S_{Mt} = \frac{\beta}{M^S_{t+1}}$$

$$\Lambda^S_{Kt} = \beta E_t \left[\Lambda^S_{Mt+1} z^S_{t+1} + \Lambda^S_{Kt+1} \left(\Phi \left(\frac{I^S_{t+1}}{K^S_{t+1}} \right) - \frac{I^S_{t+1}}{K^S_{t+1}} \Phi' \left(\frac{I^S_{t+1}}{K^S_{t+1}} \right) + 1 - \delta \right) \right]$$

$$P^F_{bt} = \beta \frac{e^S_t}{e^S_{t+1}} \frac{\Lambda^S_{Mt+1}}{\Lambda^S_{Mt}} f(s|s_t)$$

$$P^G_{bt} = \beta \frac{e^S_t}{e^S_{t+1}} \frac{e^G_{t+1}}{e^G_t} \frac{\Lambda^S_{Mt+1}}{\Lambda^S_{Mt}} f(s|s_t)$$

$$P^S_{bt} = \beta \frac{\Lambda^S_{Mt+1}}{\Lambda^S_{Mt}} f(s|s_t)$$

Firms

French firms

The intermediate good firms maximize

$$\max P^F_{Yt} Y^F_t - W^F_t h^F_t - z^F_t K^F_t$$

with $Y^F_t = A^F_t K^{F\alpha}_t (\Gamma_t h^F_t)^{1-\alpha}$

First order conditions:

$$\begin{aligned} W^F_t h^F_t &= (1-\alpha) P^F_{Yt} Y^F_t \\ z^F_t K^F_t &= \alpha P^F_{Yt} Y^F_t \end{aligned}$$

The market clearing condition is

$$Y^F_t = Y^F_{Ft} + Y^F_{Gt} + Y^F_{St}$$

The final good firms maximize

$$\max P^F_t Q^F_t - P^F_{Yt} Y^F_{Ft} - e^G_t P^G_{Yt} Y^G_{FT} - e^S_t P^S_{Yt} Y^S_{FT}$$

with

$$Q^F_t = \left(\omega_1^{1-\rho} Y^{F\rho}_{Ft} + \omega_2^{1-\rho} Y^{G\rho}_{Ft} + \omega_3^{1-\rho} Y^{S\rho}_{Ft} \right)^{1/\rho}$$

The first order conditions are

$$\begin{aligned} Y^F_{Ft} &= \left(\frac{P^F_{Yt}}{P^F_t} \right)^{\frac{1}{\rho-1}} \omega_1 Q^F_t \\ Y^G_{Ft} &= \left(\frac{e^G_t P^G_{Yt}}{P^F_t} \right)^{\frac{1}{\rho-1}} \omega_2 Q^F_t \\ Y^S_{Ft} &= \left(\frac{e^S_t P^S_{Yt}}{P^F_t} \right)^{\frac{1}{\rho-1}} \omega_3 Q^F_t \end{aligned}$$

the market clearing condition is

$$Q^F_t = C^F_t + I^F_t + G^F_t$$

and the aggregate price index is

$$P_t^F = \left(\omega_1 P_{Yt}^{F\,\frac{\rho}{\rho-1}} + \omega_2 (e_t^G P_{Yt}^G)^{\frac{\rho}{\rho-1}} + \omega_3 (e_t^S P_{Yt}^S)^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}$$

German firms

The German intermediate good firms maximize

$$\max P_{Yt}^G Y_t^G - W_t^G h_t^G - z_t^G K_t^G$$

with $Y_t^G = A_t^G K_t^{G\alpha} (\Gamma_t h_t^G)^{1-\alpha}$

First order conditions:

$$W_t^G h_t^G = (1-\alpha) P_{Yt}^G Y_t^G$$

$$z_t^G K_t^G = \alpha P_{Yt}^G Y_t^G$$

The market clearing condition is

$$Y_t^G = Y_{Ft}^G + Y_{Gt}^G + Y_{St}^G$$

The final good firms maximize

$$\max P_t^G Q_t^G - \frac{P_{Yt}^F}{e_t^G} Y_{Gt}^F - P_{Yt}^G Y_{GT}^G - \frac{e_t^S}{e_t^G} P_{Yt}^S Y_{GT}^S$$

with

$$Q_t^G = \left(\omega_1^{1-\rho} Y_{Gt}^{F\,\rho} + \omega_2^{1-\rho} Y_{Gt}^{G\,\rho} + \omega_3^{1-\rho} Y_{Gt}^{S\,\rho}\right)^{1/\rho}$$

First order conditions

$$Y_{Gt}^F = \left(\frac{P_{Yt}^F}{e_t^G P_t^G}\right)^{\frac{1}{\rho-1}} \omega_2 Q_t^G$$

$$Y_{Gt}^G = \left(\frac{P_{Yt}^G}{P_t^G}\right)^{\frac{1}{\rho-1}} \omega_1 Q_t^G$$

$$Y_{Gt}^S = \left(\frac{e_t^S P_{Yt}^S}{e_t^G P_t^G}\right)^{\frac{1}{\rho-1}} \omega_3 Q_t^G$$

the market clearing condition is

$$Q_t^G = C_t^G + I_t^G + G_t^G$$

and the aggregate price index is

$$P_t^G = \left(\omega_2 \left(\frac{P_{Yt}^F}{e_t^G} \right)^{\frac{\rho}{\rho-1}} + \omega_1 \; P_{Yt}^{G \, \frac{\rho}{\rho-1}} + \omega_3 \left(\frac{e_t^S P_{Yt}^S}{e_t^G} \right)^{\frac{\rho}{\rho-1}} \right)^{\frac{\rho-1}{\rho}}$$

UK firms

The UK intermediate good firms maximize

$$\max P_{Yt}^S Y_t^S - W_t^S h_t^S - z_t^S K_t^S$$

with $Y_t^S = A_t^S K_t^{S\alpha} (\Gamma_t h_t^S)^{1-\alpha}$

First order conditions:

$$W_t^S h_t^S = (1-\alpha) P_{Yt}^S Y_t^S$$

$$z_t^S K_t^S = \alpha P_{Yt}^S Y_t^S$$

The market clearing condition is

$$Y_t^S = Y_{Ft}^S + Y_{Gt}^S + Y_{St}^S$$

The final good firms maximize

$$\max P_t^S Q_t^S - \frac{P_{Yt}^F}{e_t^S} Y_{St}^F - \frac{e_t^G}{e_t^S} P_{Yt}^G Y_{ST}^G - P_{Yt}^S Y_{ST}^S$$

with

$$Q_t^S = \left(\omega_4^{1-\rho} Y_{St}^{F\rho} + \omega_5^{1-\rho} Y_{St}^{G\rho} + \omega_6^{1-\rho} Y_{St}^{S\rho} \right)^{1/\rho}$$

First order conditions

$$Y_{St}^F = \left(\frac{P_{Yt}^F}{e_t^S P_t^S} \right)^{\frac{1}{\rho-1}} \omega_4 Q_t^S$$

$$Y_{St}^{G} = \left(\frac{e_t^G P_{Yt}^G}{e_t^S P_t^S}\right)^{\frac{1}{\rho-1}} \omega_5 Q_t^S$$

$$Y_{St}^{S} = \left(\frac{P_{Yt}^S}{P_t^S}\right)^{\frac{1}{\rho-1}} \omega_6 Q_t^S$$

the market clearing condition is

$$Q_t^S = C_t^S + I_t^S + G_t^S$$

and the aggregate price index is

$$P_t^S = \left(\omega_4 \left(\frac{P_{Yt}^F}{e_t^S}\right)^{\frac{\rho}{\rho-1}} + \omega_5 \left(\frac{e_t^G P_{Yt}^G}{e_t^S}\right)^{\frac{\rho}{\rho-1}} + \omega_6 P_{Yt}^{S\,\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}$$

Governments

$$\text{French} \;:\; P_t^F G_t^F = P_t^F T_t^F$$

$$\text{German} \;:\; P_t^G G_t^G = P_t^G T_t^G$$

$$\text{UK} \;:\; P_t^S G_t^S = P_t^S T_t^S$$

Monetary authorities

The Taylor rules are expressed in terms of deviations from their target. For instance, $\widehat{R}_t^F$ represents $\log(R_t^f) - \log(R^f)$. Note that R is the gross interest rate.

$$\text{French} \;:\; M_{t+1}^F = \mu_t^F M_t^F$$

$$\widehat{R}_t^F = \rho^F \widehat{R}_{t-1}^F + (1-\rho^F)\left(\kappa_y^F E_t \widehat{y}_{t+1}^F + \kappa_\pi^F E_t \widehat{\pi}_{t+1}^F\right) + \zeta_t^F$$

$$\text{German} \;:\; M_{t+1}^G = \mu_t^G M_t^G$$

$$\widehat{R}_t^G = \rho^G \widehat{R}_{t-1}^G + (1-\rho^G)\left(\kappa_y^G E_t \widehat{y}_{t+1}^G + \kappa_\pi^G E_t \widehat{\pi}_{t+1}^G\right) + \zeta_t^G$$

$$\text{UK} \quad : \quad M_{t+1}^S = \mu_t^S M_t^S$$

$$\widehat{R}_t^S = \rho^S \widehat{R}_{t-1}^S + (1-\rho^S)\left(\kappa_y^S E_t \widehat{y}_{t+1}^S + \kappa_\pi^S E_t \widehat{\pi}_{t+1}^S\right) + \zeta_t^S$$

Financial markets equilibrium

$$P_{bt}^F = \frac{\Lambda_{Mt+1}^F}{\Lambda_{Mt}^F} = \frac{e_t^G}{e_{t+1}^G}\frac{\Lambda_{Mt+1}^G}{\Lambda_{Mt}^G} = \frac{e_t^S}{e_{t+1}^S}\frac{\Lambda_{Mt+1}^S}{\Lambda_{Mt}^S}$$

$$P_{bt}^G = \frac{e_{t+1}^G}{e_t^G}\frac{\Lambda_{Mt+1}^F}{\Lambda_{Mt}^F} = \frac{\Lambda_{Mt+1}^G}{\Lambda_{Mt}^G} = \frac{e_t^S}{e_{t+1}^S}\frac{e_{t+1}^G}{e_t^G}\frac{\Lambda_{Mt+1}^S}{\Lambda_{Mt}^S}$$

$$P_{bt}^S = \frac{e_{t+1}^S}{e_t^S}\frac{\Lambda_{Mt+1}^F}{\Lambda_{Mt}^F} = \frac{e_t^G}{e_{t+1}^G}\frac{e_{t+1}^S}{e_t^S}\frac{\Lambda_{Mt+1}^G}{\Lambda_{Mt}^G} = \frac{\Lambda_{Mt+1}^S}{\Lambda_{Mt}^S}$$

this implies

$$\Lambda_{Mt}^F = \eta_G \frac{\Lambda_{Mt}^G}{e_t^G} = \eta_S \frac{\Lambda_{Mt}^S}{e_t^S}$$

2.2 The equilibrium

We now turn to the description of the equilibrium of the economy. Recall that capital is perfectly mobile across countries while labor is not.

definition 1 *An equilibrium of this economy is a sequence of prices*

$$\{\mathcal{P}_t\}_{t=0}^{\infty} = \left\{W_t^j, z_t^j, P_t^j, P_{Yt}^j, p_{bt}^j(s'), R_t^j, e_t^G, e_t^S\right\} t = 0 j \in (F,G,S)^{\infty}$$

and a sequence of quantities

$$\{\mathcal{Q}_t\}_{t=0}^{\infty} = \left\{\left\{\mathcal{Q}_t^H\right\}_{t=0}^{\infty}, \left\{\mathcal{Q}_t^F\right\}_{t=0}^{\infty}\right\}$$

with

$$\left\{\mathcal{Q}_t^H\right\}_{t=0}^{\infty} = \left\{\left\{C_t^j, I_t^j, \left\{B_{it+1}^j\right\}_{i\in(F,G,S)}, K_{t+1}^j, M_{t+1}^j\right\}_{j\in(F,G,S)}\right\}_{t=0}^{\infty}$$

and

$$\left\{\mathcal{Q}_t^F\right\}_{t=0}^{\infty} = \left\{\left\{K_t^j, h_t^j, Y_t^j, \left\{Y_i^j\right\}_{i\in(F,G,S)}, Q_t^j\right\}_{j\in(F,G,S)}\right\}_{t=0}^{\infty}$$

such that:

(i) given a sequence of prices $\{\mathcal{P}_t\}_{t=0}^{\infty}$ *and a sequence of shocks,* $\{\mathcal{Q}_t^H\}_{t=0}^{\infty}$ *is a solution to the representative household's problem;*

(ii) given a sequence of prices $\{\mathcal{P}_t\}_{t=0}^{\infty}$ *and a sequence of shocks,* $\{\mathcal{Q}_t^F\}_{t=0}^{\infty}$ *is a solution to the representative firms' problem;*

(iii) given a sequence of quantities $\{\mathcal{Q}_t\}_{t=0}^{\infty}$ *and a sequence of shocks,* $\{\mathcal{P}_t\}_{t=0}^{\infty}$ *clears the goods markets in the sense*

$$Q_t^F = C_t^F + I_t^F + G_t^F \quad (17)$$

$$Q_t^G = C_t^G + I_t^G + G_t^G \quad (18)$$

$$Q_t^S = C_t^S + I_t^S + G_t^S \quad (19)$$

$$Y_t^F = Y_{Ft}^F + Y_{Gt}^F + Y_{St}^F \quad (20)$$

$$Y_t^G = Y_{Ft}^G + Y_{Gt}^G + Y_{St}^G \quad (21)$$

$$Y_t^S = Y_{Ft}^S + Y_{Gt}^S + Y_{St}^S \quad (22)$$

as well as the financial, money and capital markets.

(iv) Nominal wages are set using labor contracts of the form $W_t^j = (1-\vartheta)\widetilde{W}_t^j + \vartheta E_{t-1}\widetilde{W}_t^j$ *where* $\widetilde{W}_t^j$ *is the nominal wage that would clear the labor market in a Walrasian framework, and* $0 \leqslant \vartheta \leqslant 1$ *is the share of labor contracts in the economy.*

2.3 Numerical solution

Solving the model involves 4 steps

1. Adjusting the variables for both technological progress and nominal growth (that is, making the model stationary);
2. Calculating the deterministic steady state;
3. Log–linearizing the system around the steady state;
4. Solving the resulting dynamic system.

2.3.1 Stationarity

Let us define

$$\lambda^F_{mt} = \Lambda^F_{Mt} M^F_t \qquad \lambda^G_{mt} = \Lambda^G_{Mt} M^F_t / e^G_t \qquad \lambda^S_{mt} = \Lambda^S_{Mt} M^F_t / e^S_t$$

$$w^F_t = W^F_t / M^F_t \qquad w^G_t = e^G_{t-1} W^G_t / M^F_t \qquad w^S_t = e^S_{t-1} W^S_t / M^F_t$$

$$p^F_t = \Gamma_t P^F_t / M^F_t \qquad p^G_t = e^G_t \Gamma_t P^G_t / M^F_t \qquad p^S_t = e^S_t \Gamma_t P^S_t / M^F_t$$

$$p^F_{Yt} = \Gamma_t P^F_{Yt} / M^F_t \qquad p^G_{Yt} = e^G_t \Gamma_t P^G_{Yt} / M^F_t \qquad p^S_{Yt} = e^S_t \Gamma_t P^S_{Yt} / M^F_t$$

and

$$\Delta e^G_t = e^G_t / e^G_{t-1} \qquad \Delta e^S_t = e^S_t / e^S_{t-1}$$

$$m^G_t = e^G_{t-1} M^G_t / M^F_t \quad m^S_t = e^S_{t-1} M^S_t / M^F_t$$

Finally, we set $\lambda^j_{kt} = \Lambda^j_{Kt}\Gamma_t$ and $x^j_t = X^j_t/\Gamma_t$ for $x \in \{c, y, y^\ell_j, i, g, q, k\}$ where $\ell, j \in \{F, G, S\}$.

Note that p^j_{Bt} is the price of an asset that pays one unit in the next period. Hence, $1/p^j_{Bt}$ is the rate of return, which is known in time t. Any of the FOC related to the asset holding decisions can be integrated with respect to s. Note that, by the latter definition we have

$$R^j_t = \frac{1}{\int_S p^j_{Bt} s}$$

Hence, we have

$$\Lambda^j_{Mt} = \beta R^j_t E_t \Lambda^j_{Mt+1}$$

Finally, note that $\pi_t = P_t/P_{t-1}$, then using the definition of the deflated variables, we have:

$$\pi^F_t = p^F_t \mu^F_t / p^F_{t-1},\ \pi^G_t = p^G_t \mu^F_t / (p^G_{t-1} \Delta e^G_t) \text{ and } \pi^S_t = p^S_t \mu^F_t / (p^S_{t-1} \Delta e^S_t)$$

Equilibrium conditions of the stationary -deflated- model

$$y^F_t = a^F_t k^{F\alpha}_t h^{F1-\alpha}_t$$

$$y^G_t = a^G_t k^{G\alpha}_t h^{G1-\alpha}_t$$

$$y^S_t = a^S_t k^{S\alpha}_t h^{S1-\alpha}_t$$

$$p^F_t c^F_t = 1$$

$$p^G_t c^G_t = \Delta e^G_t m^G_t$$

$$p^S_t c^S_t = \Delta e^S_t m^S_t$$

$$p^F_t \lambda^F_{mt} = \Phi' \left(\frac{i^F_t}{k^F_t} \right) \lambda^F_{kt}$$

$$p^G_t \lambda^G_{mt} = \Phi' \left(\frac{i^G_t}{k^G_t} \right) \lambda^G_{kt}$$

$$p^S_t \lambda^S_{mt} = \Phi' \left(\frac{i^S_t}{k^S_t} \right) \lambda^S_{kt}$$

$$\theta = \lambda^F_{mt} w^F_t (1 - h^F_t)$$

$$\theta = \lambda_{mt}^G \Delta e_t^G w_t^G (1 - h_t^G)$$

$$\theta = \lambda_{mt}^S \Delta e_t^S w_t^S (1 - h_t^S)$$

$$\lambda_{mt}^F \mu_t^F = \beta$$

$$\lambda_{mt}^G \mu_t^G m_t^G \Delta e_t^G = \beta$$

$$\lambda_{mt}^S \mu_t^S m_t^S \Delta e_t^S = \beta$$

$$\lambda_{mt}^F = \eta_G \lambda_{mt}^G$$

$$\lambda_{mt}^F = \eta_S \lambda_{mt}^S$$

$$w_t^F h_t^F = (1 - \alpha) p_{Yt}^F y_t^F$$

$$\Delta e_t^G w_t^G h_t^G = (1 - \alpha) p_{Yt}^G y_t^G$$

$$\Delta e_t^S w_t^S h_t^S = (1 - \alpha) p_{Yt}^S y_t^S$$

$$q_t^F = c_t^F + i_t^F + g_t^F$$

$$q_t^G = c_t^G + i_t^G + g_t^G$$

$$q_t^S = c_t^S + i_t^S + g_t^S$$

$$y_t^F = y_{Ft}^F + y_{Gt}^F + y_{St}^F$$

$$y_t^G = y_{Ft}^G + y_{Gt}^G + y_{St}^G$$

$$y_t^S = y_{Ft}^S + y_{Gt}^S + y_{St}^S$$

$$y_{Ft}^{F} = \left(\frac{p_{Yt}^{F}}{p_t^{F}}\right)^{\frac{1}{\rho-1}} \omega_1 q_t^{F}$$

$$y_{Ft}^{G} = \left(\frac{p_{Yt}^{G}}{p_t^{F}}\right)^{\frac{1}{\rho-1}} \omega_2 q_t^{F}$$

$$y_{Ft}^{S} = \left(\frac{p_{Yt}^{S}}{p_t^{F}}\right)^{\frac{1}{\rho-1}} \omega_3 q_t^{F}$$

$$y_{Gt}^{F} = \left(\frac{p_{Yt}^{F}}{p_t^{G}}\right)^{\frac{1}{\rho-1}} \omega_2 q_t^{G}$$

$$y_{Gt}^{G} = \left(\frac{p_{Yt}^{G}}{p_t^{G}}\right)^{\frac{1}{\rho-1}} \omega_1 q_t^{G}$$

$$y_{Gt}^{S} = \left(\frac{p_{Yt}^{S}}{p_t^{G}}\right)^{\frac{1}{\rho-1}} \omega_3 q_t^{G}$$

$$y_{St}^{F} = \left(\frac{p_{Yt}^{F}}{p_t^{S}}\right)^{\frac{1}{\rho-1}} \omega_4 q_t^{S}$$

$$y_{St}^{G} = \left(\frac{p_{Yt}^{G}}{p_t^{S}}\right)^{\frac{1}{\rho-1}} \omega_5 q_t^{S}$$

$$y_{St}^{S} = \left(\frac{p_{Yt}^{S}}{p_t^{S}}\right)^{\frac{1}{\rho-1}} \omega_6 q_t^{S}$$

$$p_t^{F} = \left(\omega_1 {p_{Yt}^{F}}^{\frac{\rho}{\rho-1}} + \omega_2 {p_{Yt}^{G}}^{\frac{\rho}{\rho-1}} + \omega_3 {p_{Yt}^{S}}^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}$$

$$p_t^{G} = \left(\omega_2 {p_{Yt}^{F}}^{\frac{\rho}{\rho-1}} + \omega_1 \, {p_{Yt}^{G}}^{\frac{\rho}{\rho-1}} + \omega_3 {p_{Yt}^{S}}^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}$$

$$p_t^S = \left(\omega_4 p_{Yt}^{F}{}^{\frac{\rho}{\rho-1}} + \omega_5 p_{Yt}^{G}{}^{\frac{\rho}{\rho-1}} + \omega_6 P_{Yt}^{S}{}^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}$$

$$\gamma k_{t+1}^F = \Phi\left(\frac{i_t^F}{k_t^F}\right) k_t^F + (1-\delta) k_t^F$$

$$\gamma k_{t+1}^G = \Phi\left(\frac{i_t^G}{k_t^G}\right) k_t^G + (1-\delta) k_t^G$$

$$\gamma k_{t+1}^S = \Phi\left(\frac{i_t^S}{k_t^S}\right) k_t^S + (1-\delta) k_t^S$$

$$m_{t+1}^G = \mu_t^G \Delta e_t^G m_t^G / \mu_t^F$$

$$m_{t+1}^S = \mu_t^S \Delta e_t^S m_t^S / \mu_t^F$$

$$\widehat{R}_t^F = \rho^F \widehat{R}_{t-1}^F + (1-\rho^F)\left(\kappa_y^F E_t \widehat{y}_{t+1}^F + \kappa_\pi^F E_t\left[\widehat{p}_{t+1}^F - \widehat{p}_t^F + \mu_t^F\right]\right) + \zeta_t^F$$

$$\widehat{R}_t^G = \rho^G \widehat{R}_{t-1}^G + (1-\rho^G)\left(\kappa_y^G E_t \widehat{y}_{t+1}^G + \kappa_\pi^G E_t\left[\widehat{p}_{t+1}^G - \widehat{p}_t^G + \mu_t^F - \Delta e_{t+1}^G\right]\right) + \zeta_t^G$$

$$\widehat{R}_t^S = \rho^S \widehat{R}_{t-1}^S + (1-\rho^S)\left(\kappa_y^S E_t \widehat{y}_{t+1}^S + \kappa_\pi^S E_t\left[\widehat{p}_{t+1}^S - \widehat{p}_t^S + \mu_t^F - \Delta e_{t+1}^S\right]\right) + \zeta_t^S$$

$$\lambda_{Kt}^F = \frac{\beta}{\gamma} E_t\left[\lambda_{Mt+1}^F p_{Yt+1}^F \alpha \frac{y_{t+1}^F}{k_{t+1}^F} + \Lambda_{Kt+1}^S\left(\Phi\left(\frac{i_{t+1}^F}{k_{t+1}^F}\right) - \frac{i_{t+1}^F}{k_{t+1}^F}\Phi'\left(\frac{i_{t+1}^F}{k_{t+1}^F}\right) + 1 - \delta\right)\right]$$

$$\lambda_{Kt}^G = \frac{\beta}{\gamma} E_t\left[\lambda_{Mt+1}^G p_{Yt+1}^G \alpha \frac{y_{t+1}^G}{k_{t+1}^G} + \Lambda_{Kt+1}^G\left(\Phi\left(\frac{i_{t+1}^G}{k_{t+1}^G}\right) - \frac{i_{t+1}^G}{k_{t+1}^G}\Phi'\left(\frac{i_{t+1}^G}{k_{t+1}^G}\right) + 1 - \delta\right)\right]$$

$$\lambda_{Kt}^S = \frac{\beta}{\gamma} E_t\left[\lambda_{Mt+1}^S p_{Yt+1}^S \alpha \frac{y_{t+1}^S}{k_{t+1}^S} + \Lambda_{Kt+1}^S\left(\Phi\left(\frac{i_{t+1}^S}{k_{t+1}^S}\right) - \frac{i_{t+1}^S}{k_{t+1}^S}\Phi'\left(\frac{i_{t+1}^S}{k_{t+1}^S}\right) + 1 - \delta\right)\right]$$

$$\lambda_{mt}^F = \beta R_t^F E_t\left[\frac{\lambda_{mt+1}^F}{\mu_t^F}\right]$$

$$\lambda_{mt}^G = \beta R_t^G E_t \left[\frac{\Delta e_{t+1}^G \lambda_{mt+1}^G}{\mu_t^F} \right]$$

$$\lambda_{mt}^S = \beta R_t^S E_t \left[\frac{\Delta e_{t+1}^S \lambda_{mt+1}^S}{\mu_t^F} \right]$$

2.3.2 Calculation of the steady state

It is important to note that we do not need to calculate the values of all variables in the steady state, but only those that will be used in the log–linearization of the model. Some variables can be computed in a straightforward way so we will focus mostly on the more complicated variables. Note that the steady state is obtained by omitting the time subscripts and the expectations from the stationary system of equations

Namely, we know that

$\lambda_k^F = p^F \lambda_m^F$, $\lambda_k^G = p^G \lambda_m^G$ and $\lambda_k^S = p^S \lambda_m^S$

as we impose $\Phi'(\gamma + \delta - 1) = 1$ and $\Phi(\gamma + \delta - 1) = \gamma + \delta - 1$

$\lambda_m^F = \eta_G \lambda_m^G = \eta_S \lambda_m^S$

which implies $m^G = \eta^G$ and $m^S = \eta^S$

$\Delta e^G = \Delta e^S = 1$ from the money supply equation and imposing $\mu^F = \mu^G = \mu^S$

we impose $a^F = a^G = a^S = 1$

we impose $h^F = h^G = h^S = 0.35$

we then solve the system

$$(y^F) \qquad y^F = a^F {k^F}^\alpha {h^F}^{1-\alpha}$$

$$(y^G) \qquad y^G = a^G {k^G}^\alpha {h^G}^{1-\alpha}$$

$$(y^S) \qquad y^S = a^S k^{S\alpha} h^{S1-\alpha}$$

$$(c^F) \qquad p^F c^F = 1$$

$$(c^G) \qquad p^G c^G = m^G$$

$$(c^S) \qquad p^S c^S = m^S$$

$$(\theta) \qquad \theta = \lambda_m^F w^F (1 - h^F)$$

$$(w^G) \qquad \theta = \lambda_m^G w^G (1 - h^G)$$

$$(w^S) \qquad \theta = \lambda_m^S w^S (1 - h^S)$$

$$(\lambda_m^F) \qquad \lambda_m^F = \beta/\mu$$

$$(\lambda_m^G) \qquad \lambda_m^G = \beta/(\mu\eta_G)$$

$$(\lambda_m^S) \qquad \lambda_m^S = \beta/(\mu\eta_S)$$

$$(q^F) \qquad q^F = c^F + i^F + g^F$$

$$(q^G) \qquad q^G = c^G + i^G + g^G$$

$$(q^S) \qquad q^S = c^S + i^S + g^S$$

$$(y_F^F) \qquad y^F = y_F^F + y_G^F + y_S^F$$

$$(y_G^G) \qquad y^G = y_F^G + y_G^G + y_S^G$$

$$(y_S^S) \qquad y^S = y_F^S + y_G^S + y_S^S$$

$$(p_Y^F) \qquad y_F^F = \left(\frac{p_Y^F}{p^F}\right)^{\frac{1}{\rho-1}} \omega_1 q^F$$

$$(y_F^G) \qquad y_F^G = \left(\frac{p_Y^G}{p^F}\right)^{\frac{1}{\rho-1}} \omega_2 q^F$$

$$(y_F^S) \qquad y_F^S = \left(\frac{p_Y^S}{p^F}\right)^{\frac{1}{\rho-1}} \omega_3 q^F$$

$$(y_G^F) \qquad y_G^F = \left(\frac{p_Y^F}{p^G}\right)^{\frac{1}{\rho-1}} \omega_2 q^G$$

$$(p_Y^G) \qquad y_G^G = \left(\frac{p_Y^G}{p^G}\right)^{\frac{1}{\rho-1}} \omega_1 q^G$$

$$(y_G^S) \qquad y_G^S = \left(\frac{p_Y^S}{p^G}\right)^{\frac{1}{\rho-1}} \omega_3 q^G$$

$$(y_S^F) \qquad y_S^F = \left(\frac{p_Y^F}{p^S}\right)^{\frac{1}{\rho-1}} \omega_4 q^S$$

$$(y_S^G) \qquad y_S^G = \left(\frac{p_Y^G}{p^S}\right)^{\frac{1}{\rho-1}} \omega_5 q^S$$

$$(p_Y^S) \qquad y_S^S = \left(\frac{p_Y^S}{p^S}\right)^{\frac{1}{\rho-1}} \omega_6 q^S$$

$$(p^F) \qquad p^F = \left(\omega_1 p_Y^{F\,\frac{\rho}{\rho-1}} + \omega_2 p_Y^{G\,\frac{\rho}{\rho-1}} + \omega_3 p_Y^{S\,\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}$$

$$(p^G) \qquad p^G = \left(\omega_2 p_Y^{F\,\frac{\rho}{\rho-1}} + \omega_1 \; p_Y^{G\,\frac{\rho}{\rho-1}} + \omega_3 p_Y^{S\,\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}$$

$$(p^S) \qquad p^S = \left(\omega_4 p_Y^{F\,\frac{\rho}{\rho-1}} + \omega_5 p_Y^{G\,\frac{\rho}{\rho-1}} + \omega_6 P_Y^{S\,\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}}$$

$$(i^F) \qquad (\gamma+\delta-1)k^F = i^F$$

$$(i^G) \qquad (\gamma+\delta-1)k^G = i^G$$

$$(i^S) \qquad (\gamma+\delta-1)k^S = i^S$$

$$(k^F) \qquad (\gamma-\beta(1-\delta))p^F k^F = \beta\alpha p_Y^F y^F$$

$$(k^G) \qquad (\gamma-\beta(1-\delta))p^G k^G = \beta\alpha p_Y^G y^G$$

$$(k^S) \qquad (\gamma-\beta(1-\delta))p^S k^S = \beta\alpha p_Y^S y^S$$

$$(R) \qquad R^F = R^G = R^S = \mu/\beta$$

2.3.3 Log–linearization of the system

Each of the first order conditions can be written as :

$$E_t\varphi(x) = 0$$

where x is a vector of control and state variables, dated according to the case in t or $t+1$. Let us denote the steady state value of x by $x^\star$. By definition, $\varphi(.)$ satisfies:

$$\varphi(x^\star) = 0$$

The log–linear approximation of $\varphi(.)$ is then given by :

$$E_t\varphi(x) = E_t\left\{\varphi(x^\star) + \sum_i \left(\frac{\partial\varphi(x)}{\partial\log(x_i)}\right)\bigg|_{x=x^\star} (\log(x_i) - \log(x_i^\star)) + O(\|x\|^2)\right\}$$

where $O(\|x\|^2)$ converges to zero in probability.

Let $\widehat{x}_i$ be the percentage deviation of x_i relative to its steady state value $x_i^\star$, $\widehat{x}_i$ is thus defined by :

$$\widehat{x}_i = \log\left(\frac{x_i}{x_i^\star}\right) \simeq \frac{x_i - x_i^\star}{x_i^\star}$$

Since $\widehat{x}_i$ can be interpreted as the percentage deviation of x_i from its steady state, its coefficient can be interpreted as an elasticity.

Using the fact that $\varphi(x^\star) = 0$ and neglecting the terms of order greater or equal than 2, the log–linear approximation is given by :

$$E_t\varphi(x) \simeq E_t\left\{\sum_i \left(\frac{\partial\varphi(x)}{\partial\log(x_i)}\right)\bigg|_{x=x^\star}\widehat{x}_i\right\}$$

2.3.4 The log-linearized economies

We will now describe the log-linearized economies under a flexible exchange rate system. We log-linearized the stationary equilibrium conditions around the steady state according to the following first-order Taylor approximation $f(x_t) \approx x^* f'(x^*)\widehat{x}_t$.

$$\widehat{y}_t^F - (1-\alpha)\widehat{h}_t^F = \widehat{a}_t^F + \alpha\widehat{k}_t^F \tag{23}$$

$$\widehat{y}_t^G - (1-\alpha)\widehat{h}_t^G = \widehat{a}_t^G + \alpha\widehat{k}_t^G \tag{24}$$

$$\widehat{y}_t^S - (1-\alpha)\widehat{h}_t^S = \widehat{a}_t^S + \alpha\widehat{k}_t^S \tag{25}$$

$$\widehat{p}_t^F + \widehat{c}_t^F = 0 \tag{26}$$

$$\widehat{p}_t^G + \widehat{c}_t^G = \widehat{\triangle e}_t^G + \widehat{m}_t^G \tag{27}$$

$$\widehat{p}_t^S + \widehat{c}_t^S = \widehat{\triangle e}_t^S + \widehat{m}_t^S \tag{28}$$

$$\widehat{p}_t^F + \widehat{\lambda}_{mt}^F = \varphi(\widehat{i}_t^F - \widehat{k}_t^F) + \widehat{\lambda}_{kt}^F \tag{29}$$

$$\widehat{p}_t^G + \widehat{\lambda}_{mt}^G = \varphi(\widehat{i}_t^G - \widehat{k}_t^G) + \widehat{\lambda}_{kt}^G \tag{30}$$

$$\widehat{p}_t^S + \widehat{\lambda}_{mt}^S = \varphi(\widehat{i}_t^S - \widehat{k}_t^S) + \widehat{\lambda}_{kt}^S \tag{31}$$

$$\frac{h^F}{(1-h^F)}\widehat{h}_t^F - \widehat{w}_t^F = \widehat{\lambda}_{mt}^F \tag{32}$$

$$\frac{h^G}{(1-h^G)}\widehat{h}_t^G - \widehat{w}_t^G - \widehat{\lambda}_{mt}^G = \widehat{\triangle e}_t^G \tag{33}$$

$$\frac{h^S}{(1-h^S)}h_t^S - w_t^S - \lambda_{mt}^S = \triangle e_t^S \tag{34}$$

$$\widehat{\lambda}_{mt}^F + \widehat{\mu}_t^F = 0 \tag{35}$$

$$\widehat{\lambda}_{mt}^G + \widehat{\mu}_t^G + \widehat{\triangle e}_t^G = -\widehat{m}_t^G \tag{36}$$

$$\widehat{\lambda}_{mt}^S + \widehat{\mu}_t^S + \widehat{\triangle e}_t^S = -\widehat{m}_t^S \tag{37}$$

$$\widehat{\lambda}_{mt}^G = \widehat{\lambda}_{mt}^F \tag{38}$$

$$\widehat{\lambda}_{mt}^S = \widehat{\lambda}_{mt}^F \tag{39}$$

$$\widehat{w}_t^F + \widehat{h}_t^F - \widehat{p}_{yt}^F - \widehat{y}_t^F = 0 \tag{40}$$

$$\widehat{w}_t^G + \widehat{h}_t^G - \widehat{p}_{yt}^G - \widehat{y}_t^G = -\widehat{\triangle e}_t^G \tag{41}$$

$$\widehat{w}_t^S + \widehat{h}_t^S - \widehat{p}_{yt}^S - \widehat{y}_t^S = -\widehat{\triangle e}_t^S \tag{42}$$

$$q^F\widehat{q}_t^F - c^F\widehat{c}_t^F - i^F\widehat{i}_t^F - g^F\widehat{g}_t^F = 0 \tag{43}$$

$$q^G\widehat{q}_t^G - c^G\widehat{c}_t^G - i^G\widehat{i}_t^G - g^G\widehat{g}_t^G = 0 \tag{44}$$

$$q^S\widehat{q}_t^S - c^S\widehat{c}_t^S - i^S\widehat{i}_t^S - g^S\widehat{g}_t^S = 0 \tag{45}$$

$$y^F\widehat{y}_t^F - y_F^F\widehat{y}_{Ft}^F - y_G^F\widehat{y}_{Gt}^F - y_S^F\widehat{y}_{St}^F = 0 \tag{46}$$

$$y^G\widehat{y}_t^G - y_F^G\widehat{y}_{Ft}^G - y_G^G\widehat{y}_{Gt}^G - y_S^G\widehat{y}_{St}^G = 0 \tag{47}$$

$$y^S\widehat{y}_t^S - y_F^S\widehat{y}_{Ft}^S - y_G^S\widehat{y}_{Gt}^S - y_S^S\widehat{y}_{St}^S = 0 \tag{48}$$

$$\widehat{y}_{Ft}^F + \frac{1}{\rho-1}\widehat{p}_t^F - \frac{1}{\rho-1}\widehat{p}_{yt}^F - \widehat{q}_t^F = 0 \tag{49}$$

$$\widehat{y}_{Ft}^G + \frac{1}{\rho-1}\widehat{p}_t^F - \frac{1}{\rho-1}\widehat{p}_{yt}^G - \widehat{q}_t^F = 0 \tag{50}$$

$$\widehat{y}_{Ft}^S + \frac{1}{\rho-1}\widehat{p}_t^F - \frac{1}{\rho-1}\widehat{p}_{yt}^S - \widehat{q}_t^F = 0 \tag{51}$$

$$\widehat{y}_{Gt}^F + \frac{1}{\rho-1}\widehat{p}_t^G - \frac{1}{\rho-1}\widehat{p}_{yt}^F - \widehat{q}_t^G = 0 \tag{52}$$

$$\widehat{y}_{Gt}^G + \frac{1}{\rho-1}\widehat{p}_t^G - \frac{1}{\rho-1}\widehat{p}_{yt}^G - \widehat{q}_t^G = 0 \tag{53}$$

$$\widehat{y}_{Gt}^S + \frac{1}{\rho-1}\widehat{p}_t^G - \frac{1}{\rho-1}\widehat{p}_{yt}^S - \widehat{q}_t^G = 0 \tag{54}$$

$$\widehat{y}_{St}^F + \frac{1}{\rho-1}\widehat{p}_t^S - \frac{1}{\rho-1}\widehat{p}_{yt}^F - \widehat{q}_t^S = 0 \tag{55}$$

$$\widehat{y}_{St}^G + \frac{1}{\rho-1}\widehat{p}_t^S - \frac{1}{\rho-1}\widehat{p}_{yt}^G - \widehat{q}_t^S = 0 \tag{56}$$

$$\widehat{y}_{St}^{S}+\frac{1}{\rho-1}\widehat{p}_{t}^{S}-\frac{1}{\rho-1}\widehat{p}_{yt}^{S}-\widehat{q}_{t}^{S}=0 \tag{57}$$

$$(p^{F})^{\frac{\rho}{\rho-1}}\widehat{p}_{t}^{F}-w_{1}(p_{y}^{F})^{\frac{\rho}{\rho-1}}\widehat{p}_{yt}^{F}-w_{2}(p_{y}^{G})^{\frac{\rho}{\rho-1}}\widehat{p}_{yt}^{G}-w_{3}(p_{y}^{S})^{\frac{\rho}{\rho-1}}\widehat{p}_{yt}^{S}=0 \tag{58}$$

$$(p^{G})^{\frac{\rho}{\rho-1}}\widehat{p}_{t}^{G}-w_{2}(p_{y}^{F})^{\frac{\rho}{\rho-1}}\widehat{p}_{yt}^{F}-w_{1}(p_{y}^{G})^{\frac{\rho}{\rho-1}}\widehat{p}_{yt}^{G}-w_{3}(p_{y}^{S})^{\frac{\rho}{\rho-1}}\widehat{p}_{yt}^{S}=0 \tag{59}$$

$$(p^{S})^{\frac{\rho}{\rho-1}}\widehat{p}_{t}^{S}-w_{4}(p_{y}^{F})^{\frac{\rho}{\rho-1}}\widehat{p}_{yt}^{F}-w_{5}(p_{y}^{G})^{\frac{\rho}{\rho-1}}\widehat{p}_{yt}^{G}-w_{6}(p_{y}^{S})^{\frac{\rho}{\rho-1}}\widehat{p}_{yt}^{S}=0 \tag{60}$$

$$\gamma\widehat{k}_{t+1}^{F}+(\delta-1)\widehat{k}_{t}^{F}=(\gamma+\delta-1)\widehat{i}_{t}^{F} \tag{61}$$

$$\gamma\widehat{k}_{t+1}^{G}+(\delta-1)\widehat{k}_{t}^{G}=(\gamma+\delta-1)\widehat{i}_{t}^{G} \tag{62}$$

$$\gamma\widehat{k}_{t+1}^{S}+(\delta-1)\widehat{k}_{t}^{S}=(\gamma+\delta-1)\widehat{i}_{t}^{S} \tag{63}$$

$$\widehat{m}_{t+1}^{G}-\widehat{m}_{t}^{G}=\widehat{\mu}_{t}^{G}-\widehat{\mu}_{t}^{F}+\widehat{\triangle e}_{t}^{G} \tag{64}$$

$$\widehat{m}_{t+1}^{S}-\widehat{m}_{t}^{S}=\widehat{\mu}_{t}^{S}-\widehat{\mu}_{t}^{F}+\widehat{\triangle e}_{t}^{S} \tag{65}$$

$$\widehat{R}_{t}^{F}=\rho^{F}\widehat{R}_{t-1}^{F}+(1-\rho^{F})\kappa_{y}^{F}\widehat{y}_{t+1}^{F}+(1-\rho^{F})\kappa_{\pi}^{F}(\widehat{p}_{t+1}^{F}-\widehat{p}_{t}^{F}+\widehat{\mu}_{t}^{F})+\zeta_{t}^{F} \tag{66}$$

$$\widehat{R}_{t}^{G}=\rho^{G}\widehat{R}_{t-1}^{G}+(1-\rho^{G})\kappa_{y}^{G}\widehat{y}_{t+1}^{G}+(1-\rho^{G})\kappa_{\pi}^{G}(\widehat{p}_{t+1}^{G}-\widehat{p}_{t}^{G}+\widehat{\mu}_{t}^{F}-\widehat{\triangle e}_{t+1}^{G})+\zeta_{t}^{G} \tag{67}$$

$$\widehat{R}_{t}^{S}=\rho^{S}\widehat{R}_{t-1}^{S}+(1-\rho^{S})\kappa_{y}^{S}\widehat{y}_{t+1}^{S}+(1-\rho^{S})\kappa_{\pi}^{S}(\widehat{p}_{t+1}^{S}-\widehat{p}_{t}^{S}+\widehat{\mu}_{t}^{F}-\widehat{\triangle e}_{t+1}^{S})+\zeta_{t}^{S} \tag{68}$$

$$\begin{aligned}&\widehat{\lambda}_{kt}^{F}-\frac{\alpha\beta}{\gamma}\frac{p_{y}^{F}}{p^{F}}\frac{y^{F}}{k^{F}}\widehat{\lambda}_{mt+1}^{F}-\frac{\beta}{\gamma}(1-\delta)\widehat{\lambda}_{kt+1}^{F}+\frac{\beta}{\gamma}[\alpha\frac{p_{y}^{F}}{p^{F}}\frac{y^{F}}{k^{F}}-\varphi(\gamma+\delta-1)]\widehat{k}_{t+1}^{F}\\ =\;&\frac{\alpha\beta}{\gamma}\frac{p_{y}^{F}}{p^{F}}\frac{y^{F}}{k^{F}}\widehat{y}_{t+1}^{F}+\frac{\alpha\beta}{\gamma}\frac{p_{y}^{F}}{p^{F}}\frac{y^{F}}{k^{F}}\widehat{p}_{yt+1}^{F}-\frac{\beta}{\gamma}\varphi(\gamma+\delta-1)\widehat{i}_{t+1}^{F}\end{aligned} \tag{69}$$

$$\widehat{\lambda}_{kt}^{G} - \frac{\alpha\beta}{\gamma}\frac{p_y^G}{p^G}\frac{y^G}{k^G}\widehat{\lambda}_{mt+1}^{G} - \frac{\beta}{\gamma}(1-\delta)\widehat{\lambda}_{kt+1}^{G} + \frac{\beta}{\gamma}[\alpha\frac{p_y^G}{p^G}\frac{y^G}{k^G} - \varphi(\gamma+\delta-1)]\widehat{k}_{t+1}^{G}$$

$$= \frac{\alpha\beta}{\gamma}\frac{p_y^G}{p^G}\frac{y^G}{k^G}\widehat{y}_{t+1}^{G} + \frac{\alpha\beta}{\gamma}\frac{p_y^G}{p^G}\frac{y^G}{k^G}\widehat{p}_{yt+1}^{G} - \frac{\beta}{\gamma}\varphi(\gamma+\delta-1)\widehat{i}_{t+1}^{G} \tag{70}$$

$$\widehat{\lambda}_{kt}^{S} - \frac{\alpha\beta}{\gamma}\frac{p_y^S}{p^S}\frac{y^S}{k^S}\widehat{\lambda}_{mt+1}^{S} - \frac{\beta}{\gamma}(1-\delta)\widehat{\lambda}_{kt+1}^{S} + \frac{\beta}{\gamma}[\alpha\frac{p_y^S}{p^S}\frac{y^S}{k^S} - \varphi(\gamma+\delta-1)]\widehat{k}_{t+1}^{S}$$

$$= \frac{\alpha\beta}{\gamma}\frac{p_y^S}{p^S}\frac{y^S}{k^S}\widehat{y}_{t+1}^{S} + \frac{\alpha\beta}{\gamma}\frac{p_y^S}{p^S}\frac{y^S}{k^S}\widehat{p}_{yt+1}^{S} - \frac{\beta}{\gamma}\varphi(\gamma+\delta-1)\widehat{i}_{t+1}^{S} \tag{71}$$

$$\widehat{\lambda}_{mt}^{F} - \widehat{\lambda}_{mt+1}^{F} - \widehat{R}_{t}^{F} = -\widehat{\mu}_{t}^{F} \tag{72}$$

$$\widehat{\lambda}_{mt}^{G} - \widehat{\lambda}_{mt+1}^{G} - \widehat{R}_{t}^{G} - \widehat{\triangle e}_{t+1}^{G} = -\widehat{\mu}_{t}^{F} \tag{73}$$

$$\widehat{\lambda}_{mt}^{S} - \widehat{\lambda}_{mt+1}^{S} - \widehat{R}_{t}^{S} - \widehat{\triangle e}_{t+1}^{S} = -\widehat{\mu}_{t}^{F} \tag{74}$$

The log–linear equations can then be rewritten according to the following matrix equations :

$$M_{cc}\mathcal{C}_t = M_{cs}\mathcal{S}_t \tag{75}$$

$$M_{ss0}\mathcal{S}_{t+1} + M_{ss1}\mathcal{S}_t = M_{sc0}\mathcal{C}_{t+1} + M_{sc1}\mathcal{C}_t + M_{se}\mathcal{E}_{t+1} \tag{76}$$

Interpreting the previous system as a state–space system allows to use equation (75) as the measurement equation of the system: It links the control variables, represented by the vector $\mathcal{C}_t$, to the state variables, represented by the vector $\mathcal{S}_t$. In our problem, the control variables vector is given by :

$$\mathcal{C}_t = \left\{\widehat{c}_t^j, \widehat{i}_t^j, \widehat{h}_t^j, \widehat{y}_t^j, \widehat{q}_t^j, \widehat{y}_{\ell t}^j, \widehat{p}_t^j, \widehat{p}_{yt}^j, \widehat{w}_t^j, \widehat{\mu}_t^j\right\}'_{j\in\{F,G,S\}}$$

whereas the state variables vector is given by :

$$\mathcal{S}_t = \left\{\widehat{k}_t^j, \widehat{R}_t^j, \widehat{m}_t^G, \widehat{m}_t^S, \widehat{a}_t^j, \widehat{g}_t^j, \widehat{\zeta}_t^j, \widehat{\lambda}_{kt}^j, \widehat{\lambda}_{mt}^F, \Delta e_t^G, \Delta e_t^S\right\}'_{j\in\{F,G,S\}}$$

The second equation above(equation 76) is the state equation. It accounts for the dynamic link between control variables, state variables and the surprises, represented by[3] :

$$\mathcal{E}_t = \{E_t\widehat{x}_{t+1} - \widehat{x}_{t+1}\}, \text{for } x \in \{\mathcal{S}\}$$

This system is solved according to the method discussed in Blanchard and Kahn [1980] Because of the existence of nominal wage contracts, the model has to be solved twice:

1. First the model is solved under the assumption that wages are flexible (the walrasian model) in order to get a solution for the equilibrium wage

$$\widehat{w}_t^j = \sum_{j\in\{F,G,S\}} \left[\pi_k^j\widehat{k}_t^j + \pi_R^j\widehat{R}_t^j + \pi_a^j\widehat{a}_t^j + \pi_g^j\widehat{g}_t^j + \pi_\zeta^j\widehat{\zeta}_t^j\right] + \pi_m^G\widehat{m}_t^G + \pi_m^S\widehat{m}_t^S$$

2. This equation is then projected on the information set $\mathcal{I}_{t-1} = \left\{\widehat{k}_t^j, \widehat{R}_t^j, \widehat{m}_t^G, \widehat{m}_t^S, \widehat{a}_{t-1}^j,\right.$ $\left.\widehat{g}_{t-1}^j, \widehat{\zeta}_{t-1}^j\right\}_{j\in\{F,G,S\}}$ that represents the information set available to the agents at the time they sign the labor contracts. The resulting relation defines the wage processes and is imposed on the model together with the assumption that employment is demand determined (that is, the firms are assumed to be on their labor demand). The new model is then solved.

[3] $\widehat{\varepsilon}_t$ denote the vector of innovations for the exogenous shocks.

3 The unilateral peg: France pegs the FF to the DM while the UK maintains a flexible exchange rate (EMS)

If France pegs the FF/DM rate, $e_t^G = \overline{e^G}$, then $\Delta e_t^G = 1$ and French monetary policy must aim at maintaining the exchange rate fixed. Hence a Taylor rule can no longer be followed.

The following log-linear equations are affected by the change in monetary arrangement:

$$\widehat{p}_t^G + \widehat{c}_t^G = \widehat{m}_t^G \tag{77}$$

$$\frac{h^G}{(1-h^G)}\widehat{h}_t^G - \widehat{w}_t^G - \widehat{\lambda}_{mt}^G = 0 \tag{78}$$

$$\widehat{\lambda}_{mt}^G + \widehat{\mu}_t^G = -\widehat{m}_t^G \tag{79}$$

$$\widehat{w}_t^G + \widehat{h}_t^G - \widehat{p}_{yt}^G - \widehat{y}_t^G = 0 \tag{80}$$

$$\widehat{m}_{t+1}^G - \widehat{m}_t^G = \widehat{\mu}_t^G - \widehat{\mu}_t^F \tag{81}$$

$$\widehat{\lambda}_{mt}^F - \widehat{\lambda}_{mt+1}^F - \widehat{R}_t^G = -\widehat{\mu}_t^F \tag{82}$$

We eliminate the French equivalent to the last equation of the previous system. The control variables vector and the state variables vector respectively are given by:

$$C_t = \{\widehat{y}_t^j, \widehat{q}_t^j, \widehat{c}_t^j, \widehat{i}_t^j, \widehat{h}_t^j, \widehat{p}_t^j, \widehat{w}_t^j, \widehat{p}_{yt}^j, \widehat{y}_{lt}^j, \widehat{\mu}_t^G, \widehat{\mu}_t^S\}_{j\in\{F,G,S\}}$$

$$S_t = \{\widehat{k}_t^j, \widehat{m}_t^G, \widehat{m}_t^S, \widehat{R}_t^G, \widehat{R}_t^S, \widehat{a}_t^j, \widehat{g}_t^j, \widehat{\zeta}_t^G, \widehat{\zeta}_t^S, \widehat{\lambda}_{kt}^j, \widehat{\lambda}_{mt}^F, \widehat{\Delta e}_t^S\}_{j\in\{F,G,S\}}$$

References Blanchard, O.J. and Kahn, C.M.,1980, The Solution of Linear Difference Models under Rational expectations, *Econometrica*, 48, 1305–1313.