Precautionary Saving over the Business Cycle*

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Abstract

We study the macroeconomic implications of time-varying precautionary saving within a general equilibrium model with both aggregate and uninsurable idiosyncratic risks. In the model, agents respond to countercyclical changes in unemployment risk by altering their buffer stock of wealth, with a direct impact on aggregate consumption. Our framework generates limited cross-sectional household heterogeneity as an equilibrium outcome, thereby making it possible to analyse the role of precautionary saving over the business cycle in an analytically tractable way. The baseline model produces a response of consumption to aggregate shocks that is much closer to the data than the comparable representative-agent economy. (*JEL* E20, E21, E32)

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1 Introduction

How important are changes in precautionary asset accumulation for the propagation of business cycle shocks? In this paper, we attempt to answer this question by constructing a tractable model of time-varying precautionary saving behaviour driven by countercyclical changes in unemployment risk. Because households are assumed to be imperfectly insured against this risk, they rationally respond to such changes by altering their buffer stock of precautionary wealth. This in turn amplifies the consumption response to aggregate shocks that affect unemployment.

Our motivation for investigating the role of precautionary saving over the business cycle is based on earlier empirical evidence, which points out to a significant role for the precautionary motive in explaining the accumulation of wealth by individuals and its variations over time. Empirical studies that focus on the cross-sectional dispersion of wealth suggest that households facing higher income risk accumulate more wealth, or consume less on average, all else equal (see, e.g., Carroll, 1994; Carroll and Samwick, 1997, 1998; Engen and Gruber, 2001). This argument has been extended to the time-series dimension by Carroll (1992), Gourinchas and Parker (2001) and Parker and Preston (2005), who argue that changes in precautionary wealth accumulation following countercyclical changes in income volatility may substantially amplify fluctuations in aggregate consumption. We take stock of their results and construct a general equilibrium model in which the strength of the precautionary motive is explicitly related to the extent of unemployment risk, the main source of income fluctuations for most households (at least at business cycle frequencies).

The present contribution is both methodological and substantive. From a methodological point of view, we exhibit a class of heterogenous-agent models with incomplete markets, borrowing constraints and both aggregate (i.e., productivity) and idiosyncratic (i.e., labour market transition) shocks than can be solved by exact cross-household aggregation and rational expectations. This approach makes it possible to derive analytical results and incorporate time-varying precautionary saving into general equilibrium analysis using simple solution methods –including linearisation and undetermined coefficient methods. We are able to do so because our model has two key features. First, it endogenously generates a cross-sectional distribution of wealth with a limited number of states –rather than the large-dimensional heterogeneity implied by most heterogenous-agent models.[1] Second, in our model a substantial

[1] As is well known, the lack of perfect cross-household insurance against individual income risk usually produces a considerable amount of household heterogeneity, because the decision (state) of every household generally depends on the entire history of shocks that this household has faced (see, e.g., Huggett, 1993;

fraction of the households does not achieve full self-insurance in equilibrium (despite precautionary wealth accumulation), and thus experiences a discontinuous drop in income and consumption when unemployment strikes. This drop being of first-order magnitude, changes in the perceived likelihood that it will occur have a correspondingly first-order impact on the intensity of the precautionary motive for accumulating assets *ex ante*. Our main theoretical result is the derivation of a (common) asset-holding rule for employed households facing incomplete insurance, possibly expressed in linear form, which explicitly connects precautionary wealth accumulation to the risk of experiencing an unemployment spell.

From a substantive point of view, our paper aims at identifying and quantifying the specific role of incomplete insurance and precautionary wealth accumulation in shaping the behaviour of aggregate consumption during a typical recession –of size equal to the average post-war NBER recession. To this purpose, we use a calibrated version of the model that matches the broad features of the cross-sectional distributions of wealth and nondurables consumption in the US economy –in addition to matching other usual quantities. For plausible parameter values, our baseline model matches the trough in aggregate consumption observed in a typical recession, while the comparable representative-agent economy largely underestimates it, especially in the first quarters of the recession. Our model also predicts an unconditional consumption volatility and a correlation with output that are much closer to the data than in the case of full insurance against unemployment shocks. Our analysis thus lends support to the view that borrowing constraints and time-varying precautionary savings substantially amplify fluctuations in aggregate consumption.

Our contribution is closely related to the analysis of incomplete-market models with aggregate shocks involving large-dimensional cross-sectional heterogeneities in income and wealth. In their pioneering contribution, Krusell and Smith (1998) computed the time-series properties of a benchmark model with incomplete markets and borrowing constraints and found market incompleteness to raise the unconditional consumption-output correlation, relative to the case of full insurance against idiosyncratic shocks. Our baseline calibration similarly produces a sizeable consumption-output correlation, as we observe in post-war US data. However, we find the strong association between aggregate consumption and output implied by incomplete insurance to be much more striking following large shocks to the labour market, such as those occurring during recessions.

Several authors have proposed tractable versions of models with uninsured labour income

Aiyagari, 1994; Krusell and Smith, 1998; and more recently Krusell et al., 2011, and Guerrieri and Lorenzoni, 2011).

risk, most often by restricting the processes followed by the underlying idiosyncratic shocks. For example, Constantinides and Duffie (1996) study the asset-pricing implications of an economy in which households are hit by repeated permanent income shocks. This approach has been generalised by Heathcote et al. (2008) to the case where households' income is affected by insurable transitory shocks, in addition to imperfectly insured permanent shocks. Toche (2005), and more recently Carroll and Toche (2011) explicitly solve for households' optimal asset-holding rule in a partial-equilibrium economy where they face the risk of permanently exiting the labour market. Guerrieri and Lorenzoni (2009) analyse precautionary saving behaviour in a model with trading frictions a la Lagos and Wright (2005), and show that agents' liquidity hoarding amplify the impact of i.i.d. (aggregate and idiosyncratic) productivity shocks. Relative to these models, ours allows for the introduction of general stochastic transitions across labour market statuses, which implies that individual income shocks are i. transitory (but persistent); ii. imperfectly insured; and iii. with a conditional distribution that depends on the aggregate state. The model being fully consistent with the flow approach to the labour market, it can be evaluated using direct evidence on the cyclical movements in labour market transition rates.

The remainder of the paper is organised as follows. The following section introduces the model. It starts by describing households' consumption-saving decisions in the face of idiosyncratic unemployment risk; it then spells out firms' optimality conditions and characterises the equilibrium. In Section 3, we introduce the parameter restrictions that make our model tractable by endogenously limiting the dimensionality of the cross-sectional distribution of wealth. Section 4 calibrates the model and compares its quantitative implications to the data, as well as to those of the comparable representative-agent model. Section 6 concludes.

2 The model

The economy is populated by a continuum of households indexed by i and uniformly distributed along the unit interval, as well as a representative firm. All households rent out labour and capital to the firm, which latter produces the unique (all-purpose) good in the economy. Markets are competitive but there are frictions in the financial markets, as we describe further below.

2.1 Households

Every household i is endowed with one unit of labour, which is supplied inelastically to the representative firm if the household is employed.[2] All households are subject to idiosyncratic changes in their labour market status between "employment" and "unemployment". Employed households earn a competitive market wage (net of social contributions), while unemployed households earn a fixed unemployment benefit $\delta^i > 0$.

We assume that households can be of two types, *impatient* and *patient*, with the former and the latter having subjective discount factors $\beta^I \in (0,1)$ and $\beta^P \in \left(\beta^I, 1\right)$, respectively. As will become clear below, patient households will end up holding a large fraction of total wealth in equilibrium, leaving the impatient with only a small fraction to self-insure against unemployment risk.[3] The introduction of patient households in our incomplete-market environment is necessary to generate a realistic level of wealth dispersion, but the equilibrium with limited cross-sectional wealth heterogeneity that we construct in Section 3 could be studied without them. Impatient households occupy the subinterval $[0, \Omega]$, $\Omega \in [0,1)$, while patient households cover the complement interval $(\Omega, 1]$.

Unemployment risk. The unemployment risk faced by individual households is summarised by two probabilities: the probability that a household who is employed at date $t-1$ becomes unemployed at date t (the job-separation probability s_t), and the probability that a household who is unemployed at date $t-1$ stays so at date t (i.e., $1 - f_t$, where f_t is the job-finding probability). The law of motion for employment is:

$$n_t = (1 - n_{t-1})\, f_t + (1 - s_t)\, n_{t-1} \tag{1}$$

One typically thinks of cyclical fluctuations in (f_t, s_t) as being ultimately driven by more fundamental shocks governing the job creation policy of the firms and the natural breakdown of existing employment relationships. For example, endogenous variations in (f_t, s_t) naturally

[2] Our model ignores both changes in the labour force participation rate and changes in hours worked per employed workers, since those play a relatively minor role in the cyclical component of total hours in the US (see, e.g., Rogerson and Shimer, 2011). Incorporating an elastic labour supply for employed workers would be a simple extensions of our baseline specification.

[3] A typical implication of models with heterogenous discount factors, borrowing constraint but *complete* markets is that the constraint binds for all impatient households in equilibrium, ultimately leading the latter hold zero or negative asset wealth (see, e.g., Becker, 1980; Becker and Foias, 1987; Kiyotaki and Moore, 1997; Iacoviello, 2005). In our model, the precautionary motive will cause impatient households to hold a small but positive amount of asset wealth, despite their subjective discount rate being lower than the interest rate.

arise in a labour market plagued by search frictions, wherein both transition rates are affected by the underlying aggregate productivity shocks. We provide a model of such a labour market in Appendix A. However, we wish to emphasise here that the key market friction leading to time-varying precautionary savings is the inability of some households to perfectly insure against such transitions, a property that does not depend on the specific modelling of the labour market being adopted. For this reason, we take those transition rates as exogenous in our baseline specification, and will ultimately extract them from the data in the quantitative implementation of the model.

Impatient households. Impatient households maximise their expected life-time utility $E_0 \sum_{t=0}^{\infty} \left(\beta^I\right)^t u^I\left(c_t^i\right)$, $i \in [0, \Omega]$, where c_t^i is consumption by household i at date t and $u^I(.)$ a period utility function satisfying $u^{I\prime}(.) > 0$ and $u^{I\prime\prime}(.) \leq 0$. We restrict the set of assets that impatient households have access to in two ways. First, we assume that they cannot issue assets contingent on their employment status but only enjoy the (partial) insurance provided by the public unemployment insurance scheme; and second, we assume that these households cannot borrow against future income.[4] Given these restrictions, the only asset that can be used to smooth out idiosyncratic labour income fluctuations are claims to the capital stock. We denote by e_t^i household's i employment status at date t, with $e_t^i = 1$ if the household is employed and zero otherwise. The budget and non-negativity constraints faced by an impatient household i are:

$$a_t^i + c_t^i = e_t^i w_t^I \left(1 - \tau_t\right) + \left(1 - e_t^i\right) \delta^I + R_t a_{t-1}^i, \tag{2}$$

$$c_t^i,\ a_t^i \geq 0. \tag{3}$$

where a_t^i is household i's holdings of claims to the capital stock at the end of date t, R_t the ex post gross return on these claims, w_t^I the real wage for impatient households (assumed to be identical across them), δ^I the unemployment benefit enjoyed by these households when unemployed, and $w_t^I \tau_t$ a contribution paid by the employed and aimed at financing the unemployment insurance scheme. The Euler condition for impatient households is:

$$u^{I\prime}\left(c_t^i\right) = \beta^I E_t \left(u^{I\prime}\left(c_{t+1}^i\right) R_{t+1}\right) + \varphi_t^i, \tag{4}$$

where φ_t^i is the Lagrange coefficient associated with the borrowing constraint $a_t^i \geq 0$, with $\varphi_t^i > 0$ if the constraint is binding and $\varphi_t^i = 0$ otherwise. Condition (4), together with

[4] In our model, the employed never wish to borrow in equilibrium, so the constraint effectively binds only for some of the unemployed. Under some conditions, the model can accomodate an endogenous borrowing limit based on limited commitment. This does not substantially affect our results.

the initial asset holdings a_{-1}^{i} and the terminal condition $\lim_{t\to\infty} E_t[\beta^{It} a_t^i u^{I\prime}(c_t^i)] = 0$, fully characterise the optimal asset holdings of impatient households.

Patient households. Patient households maximise $E_0 \sum_{t=0}^{\infty} \left(\beta^P\right)^t u^P\left(c_t^i\right)$, $i \in (\Omega, 1]$, where $\beta^P \in \left(\beta^I, 1\right)$ and $u^P(.)$ is increasing and strictly concave over $[0, \infty)$. In contrast to impatient households, patient households have complete access to asset markets –including the full set of Arrow-Debreu securities and loan contracts. As discussed by Merz (1995), Andolfatto (1996) and Hall (2009), full insurance implies that these households collectively behave like a large representative 'family' in which the family head ensures equal marginal utility of wealth for all its members –despite the fact that individuals experience heterogenous employment histories. Since consumption is the only argument in the period utility, equal marginal utility of wealth implies equal consumption, so we may write the budget constraint of the family as follows:

$$C_t^P + A_t^P = R_t A_{t-1}^P + (1-\Omega)\left(n_t w_t^P (1-\tau_t) + (1-n_t)\delta^P\right), \tag{5}$$

where C_t^P (≥ 0) and A_t^P denote the consumption and end-of-period asset holdings of the family (both of which must be divided by $1-\Omega$ to find the per-family member analogues), and w_t^P and δ^P are the real wage and unemployment benefit for patient households, respectively. The Euler condition for patient households is given by:

$$u^{P\prime}\left(\frac{C_t^P}{1-\Omega}\right) = \beta^P E_t\left(u^{P\prime}\left(\frac{C_{t+1}^P}{1-\Omega}\right) R_{t+1}\right). \tag{6}$$

This condition, together with the initial asset holdings A_{-1}^P and the terminal condition $\lim_{t\to\infty} E_t[\left(\beta^P\right)^t A_t^P u^{P\prime}\left(C_t^P/(1-\Omega)\right)] = 0$, fully characterise the optimal consumption path of patient households.

2.2 Production

The representative firm produces output, Y_t, out of capital, K_t, and the units of effective labour supplied by the households. Patient and impatient households may differ in terms of relative labour efficiency, but the two types are perfect substitutes from the point of view of the firm in terms of effective labour units. More specifically, the production function takes the form $Y_t = z_t G\left(K_t, n_t^I + \kappa n_t^P\right)$, where n_t^I and n_t^P denote the firm's use of impatient and patient households' labour input, respectively, $\kappa > 0$ the relative efficiency of patient households' labour (with the efficiency of impatient households' labour normalised to one), $\{z_t\}_{t=0}^{\infty}$ a stochastic aggregate productivity process with unconditional mean $z^* = 1$, and where

$G(.,.)$ exhibits positive, decreasing marginal products and constant returns to scale (CRS).[5] Defining $k_t \equiv K_t/\left(n_t^I + \kappa n_t^P\right)$ as capital per unit of effective labour and $g(k_t) \equiv G(k_t, 1)$ the corresponding intensive-form production function, we have $Y_t = z_t\left(n_t^I + \kappa n_t^P\right) g(k_t)$. Capital depreciates at the constant rate $\mu \in [0,1]$. Given R_t and z_t, the optimal demand for capital by the representative firm satisfies:

$$z_t g'(k_t) = R_t - 1 + \mu. \tag{7}$$

On the other hand, the optimal demands for the two labour types in a perfectly competitive labour market must satisfy $z_t G_2\left(K_t, n_t^I + \kappa n_t^P\right) = w_t^I = w_t^P/\kappa$, where w_t^I is the real wage per unit of effective labour.

2.3 Market clearing

By the law of large numbers and the fact that all households face identical transition rates in the labour market, the equilibrium numbers of impatient and patient households working in the representative firm are $n_t^I = \Omega n_t$ and $n_t^P = (1-\Omega)\, n_t$, respectively. Consequently, effective labour is $n_t^I + \kappa n_t^P = (\Omega + (1-\Omega)\kappa)\, n_t$, where n_t is given by (1). Moreover, by the CRS assumption the equilibrium real wage per unit of effective labour is $w_t^I = z_t\left(g(k_t) - k_t g'(k_t)\right)$.[6]

Let $F_t(\tilde{a}, e)$ denote the measure at date t of impatient households with beginning-of-period asset wealth $\tilde{a}$ and employment status e, and $a_t(\tilde{a}, e)$ and $c_t(\tilde{a}, e)$ the corresponding policy functions for assets and consumption, respectively.[7] Since those households are in share Ω in the economy, clearing of the market for claims to the capital stock requires that

$$A_{t-1}^P + \Omega \sum_{e=0,1} \int_{\tilde{a}=0}^{+\infty} a_{t-1}(\tilde{a}, e)\, dF_{t-1}(\tilde{a}, e) = (\Omega + (1-\Omega)\kappa)\, n_t k_t, \tag{8}$$

where the left hand side of (8) is total asset holdings by all households at the end of date $t-1$ and the right hand side the demand for capital by the representative firm at date t. Clearing

[5] As will be clear in Section 4 below, the introduction of an efficiency premium for patient households (i.e., $\kappa > 1$), which raises their labour income share in equilibrium relative to the symmetric case ($\kappa = 1$), is necessary to match the cross-sectional dispersion of consumption, for any plausible level of wealth dispersion.

[6] In a noncompetitive labour market with search frictions such as that considered in Appendix A, the real wage must lie strictly below the marginal product of labour at least at some point in time during the employment relationship in order to make it worthwhile for the firm to pay the vacancy costs.

[7] Our formulation of the market-clearing conditions (8)–(9) presumes the existence of a recursive formulation of the household's problem with $(\tilde{a}, e)$ as individual state variables, as this will be the case in the equilibrium that we are considering. See, e.g., Heathcote (2005) for a nonrecursive formulation of the household's problem.

of the goods market requires:

$$C_t^P + \Omega \sum_{e=0,1} \int_{\tilde{a}=0}^{+\infty} c_t(\tilde{a}, e)\, dF_t(\tilde{a}, e) + I_t = z_t (\Omega + (1-\Omega)\kappa)\, n_t g(k_t), \tag{9}$$

where the left hand side of (9) includes the consumption of all households as well as aggregate investment, $I_t \equiv (\Omega + (1-\Omega)\kappa)(n_{t+1}k_{t+1} - (1-\mu)n_t k_t)$, and the right hand side is output. Finally, we require the unemployment insurance scheme to be balanced, i.e.,

$$\tau_t n_t \left(\Omega w_t^I + (1-\Omega) w_t^P\right) = (1-n_t)\left(\Omega \delta^I + (1-\Omega)\delta^P\right), \tag{10}$$

where the left and right hand sides of (10) are total unemployment contributions and benefits, respectively.

An *equilibrium* of this economy is defined as a sequence of households' decisions $\{C_t^P, c_t^i, A_t^P, a_t^i\}_{t=0}^{\infty}$, $i \in [0, \Omega]$, firm's capital per effective labour unit $\{k_t\}_{t=0}^{\infty}$, and aggregate variables $\{n_t, w_t^I, R_t, \tau_t\}_{t=0}^{\infty}$, which satisfy the households' and the representative firm's optimality conditions (4), (6) and (7), together with the market-clearing and balanced-budget conditions (8)–(10), given the forcing sequences $\{f_t, s_t, z_t\}_{t=0}^{\infty}$ and the initial wealth distribution $\left(A_{-1}^p, a_{-1}^i,\right)_{i\in[0,\Omega]}$.

3 An equilibrium with limited cross-sectional heterogeneity

As is well known, dynamic general equilibrium models with incomplete markets and borrowing constraints are not tractable in general, essentially because any household's decisions depend on its accumulated asset wealth, while the latter is determined by the entire history of idiosyncratic shocks that this household has faced. In consequence, the asymptotic cross-sectional distribution of wealth usually has infinitely many states, and hence infinitely many agent types end up populating the economy (Aiyagari, 1994; Krusell and Smith, 1998). In this paper, we make specific assumptions about impatient household's period utility and the tightness of the borrowing constraint, which ensure that the cross-sectional distribution of wealth has a finite number of wealth states as an equilibrium outcome. As a result, the economy is characterized by a finite number of heterogenous agents whose behaviour can be aggregated exactly, thereby making it possible to represent the model's dynamics via a standard (small-scale) dynamic system. In the remainder of the paper, we focus on the simplest equilibrium, which involves exactly two possible wealth states for impatient households.

However, we show in Section 3.3 and Appendix B how this approach can be generalised to construct equilibria with any finite number of wealth states.

3.1 Assumptions and conjectured equilibrium

Let us first assume that the instant utility function for impatient households $u^I(c)$ is i) continuous, increasing and differentiable over $[0, +\infty)$, ii) strictly concave with local relative risk aversion coefficient $\sigma^I(c) = -cu^{I\prime\prime}(c)/u^{I\prime}(c) > 0$ over $[0, c^*]$, where c^* is an exogenous, positive threshold, and iii) linear with slope $\eta > 0$ over $(c^*, +\infty)$ (see Figure 2). Essentially, this utility function (an extreme form of decreasing relative risk aversion) implies that high-consumption (i.e., relatively wealthy) impatient households do not mind moderate consumption fluctuations –i.e., as long as the implied optimal consumption level says inside $(c^*, +\infty)$– but dislike substantial consumption drops –those that would cause consumption to fall inside the $[0, c^*]$ interval. In the equilibrium that we are focusing on, 'moderate' consumption fluctuations refer to consumption changes triggered by variations in asset and wage incomes *conditional on the household remaining employed*; in contrast, 'substantial consumption drops' refer to those triggered by the large falls in current income that are associated with a *change in employment status from employed to unemployed.* In other words, we are constructing an equilibrium in which the following condition holds:

$$\textbf{Condition 1}: \forall i \in [0, \Omega],\ e_t^i = 1 \Rightarrow c_t^i > c^*,\ e_t^i = 0 \Rightarrow c_t^i \leq c^*. \tag{11}$$

As we shall see shortly, one implication of this utility function and ranking of consumption levels is that employed households fear unemployment, and consequently engage in precautionary saving behaviour *ex ante* in order to limit (but without being able to fully eliminate) the associated rise in marginal utility.

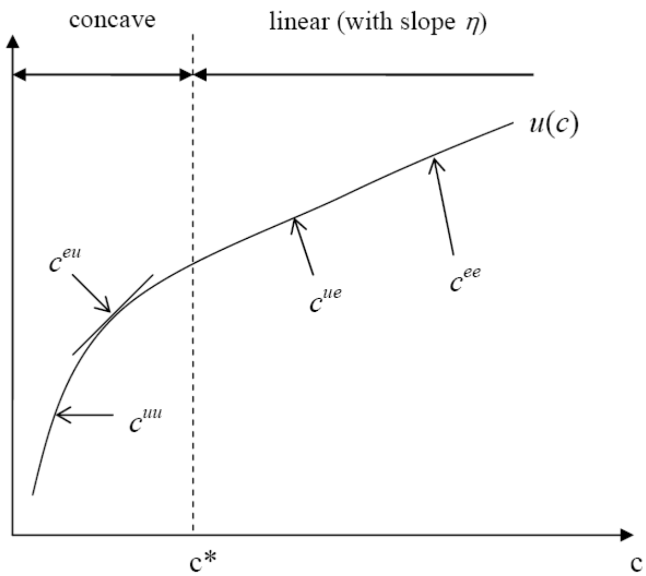


Figure 1. Instant utility function of impatient households, $u^I(c)$.

The second feature of the equilibrium that we are constructing is that the borrowing constraint in (3) is binding (that is, the Lagrange multiplier in (4) is positive) for all unemployed households, while it is never binding for employed households. In consequence, the end-of-period asset holdings of the former are zero (rather than negative), while those of the latter are strictly positive. In short, the equilibrium that we are constructing will satisfy the following condition:[8]

$$\textbf{Condition 2}: \forall i \in [0, \Omega], \begin{cases} e_t^i = 0 \Rightarrow u'(c_t^i) > E_t\left(\beta^I u'\left(c_{t+1}^i\right) R_{t+1}\right) \text{ and } a_t^i = 0, \\ e_t^i = 1 \Rightarrow u'(c_t^i) = E_t\left(\beta^I u'\left(c_{t+1}^i\right) R_{t+1}\right) \text{ and } a_t^i > 0. \end{cases} \quad (12)$$

Equations (11)–(12) have direct implications for the optimal asset holdings of employed households. By construction, a household who is employed at date t has asset wealth $a_t^i R_{t+1}$ at the beginning of date $t+1$. If the household falls into unemployment at date $t+1$, then the borrowing constraint becomes binding and the households liquidates all assets. This implies that the household enjoys consumption

$$c_{t+1}^i = \delta^I + a_t^i R_{t+1} \quad (13)$$

[8]When parameters are such that the borrowing constraint is binding for *all* impatient households including the employed (i.e., $a_t^i = 0 \; \forall i \in [0, \Omega]$), those become mere "rule-of-thumb" consumers who spend their entire disposable income on goods in every period. We consider this special case when it arises in one of our sensitivity experiments in Section 4.3.

and marginal utility $u^{I\prime}\left(\delta^I + a^i_{t+1}R_{t+1}\right)$. If follows from conditions (11)–(12) that the Euler equation characterising the asset holdings of an employed, impatient household i is given by:

$$\eta = \beta^I E_t \left[\left(\left(1 - s_{t+1}\right)\eta + s_{t+1}u^{I\prime}\left(\delta^I + a^i_t R_{t+1}\right)\right) R_{t+1}\right]. \tag{14}$$

The left hand side of (14) is the current marginal utility of this household, which is equal to η under condition (11). The left hand side of (14) is expected, discounted future marginal utility, with marginal utility at date $t+1$ being broken into the two possible employment statuses that this household may experience at that date, weighted by their probabilities of occurrence (consequently, the operator $E_t\left(.\right)$ in (14) is with respect to aggregate uncertainty only). More specifically, if the household stays employed, which occurs with probability $1 - s_{t+1}$, it enjoys marginal utility η at date $t+1$ (by equation (11)); if the household falls into unemployment, which occurs with complementary probability, it liquidates assets (by equation (12)) and, as discussed above, enjoys marginal utility $u^{I\prime}\left(\delta^I + a^i_t R_{t+1}\right)$.

Since equation (14) pins down a^i_t as a function of *aggregate* variables only (i.e., s_{t+1} and R_{t+1}), asset holdings are symmetric across employed households –and strictly positive under condition (12). We may thus write:

$$\forall i \in \left[0, \Omega\right],\ e^i_t = 1 \Rightarrow a^i_t = a_t > 0. \tag{15}$$

To summarise, under conditions (11)–(12) the cross-sectional distribution of wealth has exactly two states (0 and a_t), so that the economy is populated by exactly *four* types of impatient households –since from (2) the type of a household depends on both beginning- and end-of-period asset wealth. We call these types 'ij', $i, j = e, u$, where i (j) refers to the household's employment status in the previous (current) date (for example, a 'ue household' is currently employed but was unemployed in the previous period, and its consumption at date t is c^{ue}_t). The ranking of consumption levels for these households in the conjectured equilibrium is shown in Figure 1.

To get further insight into how unemployment risk affects precautionary asset accumulation, it may be useful to use (14)–(15) to rewrite the optimal asset holding condition for employed, impatient households as follows:

$$\beta^I E_t \left[\left(1 + s_{t+1}\left(\frac{u^{I\prime}\left(\delta^I + a_t R_{t+1}\right) - \eta}{\eta}\right)\right) R_{t+1}\right] = 1. \tag{16}$$

Consider, for the sake of the argument, the effect of a fully predictable change in s_{t+1} holding R_{t+1} constant. The direct effect is to raise $1+s_{t+1}[u^{I\prime}\left(\delta^I + a_t R_{t+1}\right) - \eta]/\eta$, since the proportional change in marginal utility associated with becoming unemployed, $[u^{I\prime}\left(\delta^I + a_t R_{t+1}\right) -$

$\eta]/\eta$, is positive (see Figure 1). Hence, $u^{I\prime}\left(\delta^I + a_t R_{t+1}\right)$ must go down for (16) to hold, which is achieved by raising date t asset holdings, a_t.

Aggregation. The limited cross-sectional heterogeneity that prevails across impatient households implies that we can exactly aggregate their asset holding choices. From (12) and (15), total asset holdings by impatient households is

$$A_t^I \equiv \Omega \sum_{e=0,1} \int_{\tilde{a}=0}^{+\infty} a_t\left(\tilde{a}, e\right) dF_t\left(\tilde{a}, e\right) = \Omega n_t a_t, \tag{17}$$

which can be substituted into the market-clearing condition (8). Similarly, aggregating the budget constraints of those households (equation (2)) under (12) and (15), we find total consumption by impatient households to be:

$$\begin{aligned} C_t^I &\equiv \Omega \sum_{e=0,1} \int_{\tilde{a}=0}^{+\infty} c_t\left(\tilde{a}, e\right) dF_t\left(\tilde{a}, e\right) \\ &= \underbrace{\Omega\left(n_t w_t^I \left(1-\tau_t\right) + \left(1-n_t\right)\delta^I\right) + \left(R_t - 1\right) A_{t-1}^I}_{\text{net income}} - \underbrace{\Omega\Delta\left(n_t a_t\right)}_{\text{change in asset wealth}}, \end{aligned} \tag{18}$$

where Δ is the difference operator.

Equation (18) summarises the determinant of total consumption by impatient households in the economy. At date t, their aggregate net income is given by past asset accumulation and current factor payments –and hence taken as given by the households in the current period. The change in their total asset holdings, $\Omega\Delta\left(n_t a_t\right)$, depends on both the change in the number of precautionary savers, Ωn_t, and the assets held by each of them, a_t. In the remainder of the paper, we also refer to Ωn_t and a_t as the "extensive" and "intensive" asset holding margins, respectively. The former is determined by employment flows is thus beyond the households' control. The latter is their key choice variable and obeys (16).

3.2 Existence conditions and steady state

Existence conditions. The equilibrium with limited cross-sectional heterogeneity described so far exists provided that three conditions are satisfied. First, the postulated ranking of consumption levels for impatient households in (11) must hold in equilibrium. Second, unemployed, impatient households must face a binding borrowing constraint (first line of (12)). And third, employed, impatient households must not face a binding borrowing constraint (second line of (12)).

- From impatient households' budget constraint (2) and under the conjectured equilibrium (in which $a_t > 0$), we have $c_t^{uu} = \delta^I < c_t^{eu} = \delta^I + a_{t-1}R_t$ and $c_t^{ee} = w_t^I (1 - \tau_t) - a_t + a_{t-1}R_t > c_t^{ue} = w_t^I (1 - \tau_t) - a_t$. Hence, a necessary and sufficient condition for (11) to hold is $c_t^{eu} < c^* < c_t^{ue}$, that is,

$$\delta^I + a_{t-1}R_t < c^* < w_t^I (1 - \tau_t) - a_t. \tag{19}$$

- Unemployed, impatient households can be of two types, uu and eu, and we require both types to face a binding borrowing constraint in equilibrium. However, since $c_t^{uu} < c_t^{eu}$ (and hence $u'(c_t^{uu}) > u'(c_t^{eu})$), a necessary and sufficient condition for both types to be constrained is $u'(c_t^{eu}) > \beta E_t((f_{t+1}u'(c_t^{ue}) + (1 - f_{t+1}) u'(c_{t+1}^{uu})) R_{t+1})$, where the right hand side of the inequality is the expected, discounted marginal utility of an eu household who is contemplating the possibility of either remaining unemployed (with probability $1 - f_{t+1}$) or finding a job (with probability f_{t+1}). Under the conjectured equilibrium we have $u'(c_t^{ue}) = \eta$ and $c_{t+1}^{uu} = \delta^I$, so the latter inequality becomes:

$$u'\left(\delta^I + a_{t-1}R_t\right) > \beta E_t\left(\left(f_{t+1}\eta + (1 - f_{t+1}) u'\left(\delta^I\right)\right) R_{t+1}\right). \tag{20}$$

- Finally, the asset holding choices of employed, impatient households will be interior (that is, the Euler equation in (12) will hold with equality) if, given the conditional distributions of R_{t+1} and s_{t+1}, the solution a_t to (16) satisfies

$$a_t > 0. \tag{21}$$

In all our quantitative experiments in Section 4, we directly check numerically that (19)–(21) do hold all along the stochastic equilibrium, given the path of the exogenous state variables under consideration. In what follows, we compute the steady state of our conjectured equilibrium and derive a set of necessary and sufficient conditions for (19)–(21) to hold in the absence of aggregate shocks. By continuity, they will also hold in the stochastic equilibrium provided that the magnitude of aggregate shocks is not too large. Finally,

Steady state. In the steady state, the real interest rate is determined by the discount rate of the most patient households, so that $R^* = 1/\beta^P$ (see (6)). From (1) and (7), the steady state levels of employment and capital per effective labour unit are

$$n^* = \frac{f^*}{f^* + s^*},\ k^* = g'^{-1}\left(\frac{1}{\beta^P} - 1 + \mu\right). \tag{22}$$

A key variable in the model is the level of asset holdings that employed, impatient households hold as a buffer against unemployment risk. From (16) and the fact that $R^* = 1/\beta^P$, its value at the steady state is

$$a^* = \beta^P \left[u^{I\prime -1} \left(\eta \left(1 + \frac{\beta^P - \beta^I}{\beta^I s^*} \right) \right) - \delta^I \right]. \tag{23}$$

Finally, from (8) and (17), steady state (total) asset holdings by patient households are $A^{P*} = n^* \left((\Omega + (1 - \Omega) \kappa) k^* - \Omega a^* \right)$. The other relevant steady state values directly follow.

We may now state the following proposition, which establishes the existence of the equilibrium with limited heterogeneity at the steady state.

Proposition 1. Assume that there are no aggregate shocks, and that

$$\begin{aligned} \delta^I &< u^{I\prime -1} \left(\eta \left(1 + \frac{\beta^P - \beta^I}{\beta^I s^*} \right) \right) \\ &< \min \left[\frac{\left(g(k^*) - k^* g'(k^*) \right) (1 - \tau^*) + \beta^P \delta^I}{1 + \beta^P}, u'^{-1} \left(\frac{\beta^I \left(f^* \eta + (1 - f^*) u^{I\prime} \left(\delta^I \right) \right)}{\beta^P} \right) \right], \end{aligned}$$

where

$$\tau^* = \left(\frac{1 - n^*}{n^*} \right) \frac{\Omega \delta^I + (1 - \Omega) \delta^P}{(\Omega + (1 - \Omega) \kappa) \left(g(k^*) - k^* g'(k^*) \right)} \tag{24}$$

and (n^*, k^*) are given by (22). Then, it is always possible to find a utility threshold c^* such that the conjectured limited-heterogeneity equilibrium described above exists.

Proof. The first inequality in Proposition 1 ensures that $a^* > 0$ in equation (23), so that (21) hold in the steady state. Evaluating (16) at the steady state, we find

$$u' \left(\delta^I + a^* R^* \right) = (\eta / s^*) \left(\beta^P / \beta^I - (1 - s^*) \right). \tag{25}$$

Substituting (25) and $w^{I*} = g(k^*) - k^* g'(k^*)$ into the steady-state counterparts of (19)–(20), we obtain the following conditions:

$$\begin{aligned} \eta \left(1 + \frac{\beta^P - \beta^I}{\beta s^*} \right) &> \frac{\beta^I}{\beta^P} \left(f^* \eta + (1 - f^*) u' \left(\delta^I \right) \right), \\ u^{I\prime -1} \left(\eta \left(1 + \frac{\beta^P - \beta^I}{\beta^I s^*} \right) \right) &< \frac{\left(g(k^*) - k^* g'(k^*) \right) (1 - \tau^*) + \beta^P \delta^I}{1 + \beta^P}, \end{aligned}$$

and hence the second inequality in the proposition. ■

The inequalities in Proposition 1 ensure that, in the steady state, i. the candidate equilibrium features exactly two possible wealth levels for impatient households (0 and $a^* > 0$); and

ii. the implied ranking of individual consumption levels is indeed such that we can "reverse-engineer" an instant utility function for these households of the form depicted in Figure 1. Those inequalities can straightforwardly be checked once specific values are assigned to the deep parameters of the model. As we argue in Section 4 below, it is satisfied for plausible such values when we calibrate the model to the US economy. The reason for which it does is as follows. Our limited-heterogeneity equilibrium requires that impatient, unemployed households be borrowing-constrained (i.e., they would like to borrow against future income but are prevented from doing so), while impatient, employed households accumulate sufficiently little wealth in equilibrium (so that this wealth be exhausted within a quarter of unemployment). In the US, the quarter-to-quarter probability of leaving unemployment is high and the replacement ratio relatively low, leading the unemployed's expected income to be sufficiently larger than current income for these households to be willing to borrow. On the other hand, the US distribution of wealth is fairly unequal, leading a large fraction of the population (the impatient in our model) to hold a very small fraction of total wealth.

Linearised asset holding rule. It is important to stress that local time-variations in the probability to become unemployed, s_{t+1}, have a *first-order* effect on precautionary asset accumulation at the individual level, a_t. This is because even without aggregate risk a change in employment status from employment to unemployment at date $t+1$ is associated with a large fall in individual consumption, and hence with a infra-marginal rise in marginal utility from η to $u^{I\prime}(c^{eu}) > \eta$.[9] The probability s_{t+1} weights this possibility in employed households' Euler equation (see (16)), so even small changes in s_{t+1} have a sizeable impact on asset holdings and consumption choices. Linearising (16) around the steady state calculated above, we find the following approximate individual asset accumulation rule:

$$a_t \simeq a^* + \Gamma_s E_t (s_{t+1} - s^*) + \Gamma_R E_t (R_{t+1} - R^*),$$

with

$$\Gamma_s = \frac{\left(\beta^P - \beta^I\right)\left(\beta^P \delta^I + a^*\right)}{\left(\beta^P - \beta^I (1 - s^*)\right) \sigma^I (c^{eu*})} > 0,$$

[9] This property distinguishes our model from those which root the precautionary motive into households' 'prudence' (Kimball, 1990). In that framework, time-variations in precautionary savings may follow from changes in the second-order term of future marginal utility (see, e.g., Gourinchas and Parker, 2001; Parker and Preston, 2005). It is apparent from (16) that a mean-preserving increase in employed households' uncertainty about future labour income taking the form of an increase in s_{t+1} (and a corresponding rise in w_{t+1} to keep expected income constant) raises asset holdings –the usual definition of 'precautionary saving.'

and where a^* is given by (23), $\sigma^I(c^{eu*}) \equiv -c^{eu*}u^{I\prime\prime}(c^{eu*})/u^{I\prime}(c^{eu*})$ and $c^{eu*} = \delta^I + a^*R^*$. The composite parameter Γ_s measures the strength of the response of individuals' precautionary wealth following predicted changes in unemployment risk, such as summarised by the period-to-period separation rate s_{t+1}.[10]

3.3 Equilibria with multiple wealth states

In the previous sections, we have constructed an equilibrium with limited cross-sectional heterogeneity characterised by the simplest (nondegenerate) distribution of wealth, that with two states. A key feature of this equilibrium is that impatient households face a binding borrowing constraint after the first period of unemployment –and hence liquidate their entire asset wealth. As we argue next, instant asset liquidation by wealth-poor households is a natural outcome of our framework when we calibrate it on a quarterly basis and using US data on the cross-sectional distribution of wealth. However, we emphasise that the same approach can be use to construct equilibria with any finite number of wealth states, wherein households gradually, rather than instantly, sell assets to offset their individual income fall. To see this intuitively, consider the steady state of the simple equilibrium described above and take its existence condition with respect to the bindingness of the borrowing constraint for an impatient households who fall into unemployment:

$$u^{I\prime}\left(\delta^I + a^*R^*\right) > \beta^I\left[\left(\eta f^* + (1-f^*)\,u^{I\prime}\left(\delta^I\right)\right)R^*\right], \tag{27}$$

where $\delta^I + a^*R^*$ is the consumption of those households under full liquidation, η their marginal utility in the next period if they exit unemployment, and δ^I their consumption in the next period if they stay unemployed (with no assets left, by construction).

The circumstances leading to the violation of inequality (27), so that the equilibrium with two wealth states described above no longer exists, are the following. First, the job-finding rate f^* or the unemployment benefit δ^I may be too low, leading to high marginal utility in the next period (the right hand side of (27)), thereby urging the household to transfer wealth into the future. Second, the asset holdings accumulated when employed (a^*, which is itself determined by (23)) may be too high, leading to low current marginal utility (the left hand side (27)), thereby making this transfer little costly to the household. However, even if inequality (27) is violated for one of these reasons, a similar condition might nevertheless hold

[10] Γ_R may be positive or negative depending on the relative strengths of the intertemporal income and substitution effects. In particular, high values of σ^I produce asset accumulation rules that prescibe an increase in a_t following a fall in $E_t(R_{t+1})$.

for households having experienced *two* consecutive periods of unemployment, because those have less wealth (and hence higher current marginal utility) than in the first unemployment period. In this case, the equilibrium will have exactly three wealth states (including two strictly positive), rather than two.

We derive in Appendix B a set of necessary and sufficient conditions for the existence and uniqueness of limited-heterogeneity equilibria with $m+1$ wealth states (that is, m strictly positive wealth states), thereby generalising the constructive approach used in Sections 3.1–3.2. As before, we focus on those conditions at the steady state, and resort to perturbation arguments to extend them to the stochastic equilibrium. An equilibrium with $m+1$ wealth states has the property that the only impatient households facing a binding borrowing constraint are those having experienced at least m consecutive periods of unemployment. Before the mth unemployment period, the asset wealth and consumption level of these households decreases gradually. From the $m+1$th unemployment period, these households face a binding borrowing constraint, hold no wealth, and have a flat consumption path (equal to δ^I). Finally, we show that an equilibrium with $m+1$ wealth states is associated with $2(m+1)$ types of impatient households (that is, the cross-sectional distribution of consumption has $2(m+1)$ possible states).

4 Time-varying precautionary saving and consumption fluctuations

The model developed above implies that some households rationally respond to countercyclical changes in unemployment risk by raising precautionary wealth –and thus by cutting down individual consumption more than they would have done without the precautionary motive. We now wish to assess the extent of this effect on aggregate consumption when realistic unemployment shocks are fed into our model economy. To this purpose, we compute the response of aggregate consumption to aggregate shocks implied by our baseline model, and then compare it with i. the data, and ii. a comparable representative-agent economy.

4.1 Model summary

We start by writing down the dynamic system summarising the behaviour of our incomplete-market economy, at the level of aggregation that is relevant for the quantitative exercises that follow. The model includes three forcing variables (z_t, f_t and s_t) and nine endogenous variables, namely: employment and capital per effective labour unit, n_t and k_t,; the total consumption of impatient and patient households, C_t^I and C_t^P; the corresponding asset levels, i.e., A_t^P for the representative family and a_t for an employed, impatient household; the factor prices R_t and w_t^I; and the unemployment contribution rate, τ_t. These endogenous variables are linked through the following equations:

$$\beta^I E_t \left[\left(1 + s_{t+1} \left(\frac{u^{I\prime}\left(\delta^I + a_t R_{t+1}\right) - \eta}{\eta} \right) \right) R_{t+1} \right] = 1 \tag{EE-I}$$

$$C_t^I + \Omega n_t a_t = \Omega \left(n_t w_t^I \left(1 - \tau_t\right) + \left(1 - n_t\right) \delta^I \right) + \Omega R_t n_{t-1} a_{t-1} \tag{BC-I}$$

$$\beta^P E_t \left(\frac{u^{P\prime}\left(C_{t+1}^P / \left(1 - \Omega\right)\right)}{u^{P\prime}\left(C_t^P / \left(1 - \Omega\right)\right)} R_{t+1} \right) = 1 \tag{EE-P}$$

$$C_t^P + A_t^P = \left(1 - \Omega\right) \left(\kappa n_t w_t^I \left(1 - \tau_t\right) + \left(1 - n_t\right) \delta^P \right) + R_t A_{t-1}^P \tag{BC-P}$$

$$R_t = z_t g'\left(k_t\right) + 1 - \mu \tag{IR}$$

$$w_t^I = z_t \left(g\left(k_t\right) - k_t g'\left(k_t\right) \right) \tag{WA}$$

$$A_{t-1}^P + \Omega n_{t-1} a_{t-1} = \left(\Omega + \left(1 - \Omega\right) \kappa \right) n_t k_t \tag{CM}$$

$$\tau_t n_t w_t^I \left(\Omega + \left(1 - \Omega\right) \kappa \right) = \left(1 - n_t\right) \left(\Omega \delta^I + \left(1 - \Omega\right) \delta^P \right) \tag{UI}$$

$$n_t = \left(1 - n_{t-1}\right) f_t + \left(1 - s_t\right) n_{t-1} \tag{EM}$$

Equations (EE-I)–(BC-I) are the Euler condition and aggregate budget constraint for impatient households (where (BC-I) is just a rewriting of (18)). Equations (EE-P)–(BC-P) are the same conditions for patient households, such as given by (5)–(6) and where w_t^P has been replaced by its equilibrium value, κw_t^I. (IR) follows from the representative firm's optimality condition (7), with the factor price frontier under CRS giving w_t^I in (WA). Equation (CM) is the market-clearing condition for capital, which follows from substituting (17) into (8). Finally, (UI) is the balanced-budget condition for the unemployment insurance scheme (where again $w_t^P = \kappa w_t^I$ has been substituted into (10)), while (EM) is the law of motion for employment. The incomplete-insurance model nests the representative-agent economy as a special case, which is obtained by setting $\Omega = 0$ (and possibly adjusting κ to hold average productivity unchanged –see below).

4.2 Parameterisation

The period is a quarter. On the production side, we assume a Cobb-Douglas production function $Y_t = z_t K_t^\alpha n_t^{1-\alpha}$, with $\alpha = 1/3$, and a depreciation rate $\mu = 2.5\%$.

On the households' side, we adopt the following baseline parameters (see Section 4.3 for the sensitivity of our results to parameter changes). We set the share of impatient households, Ω, to 0.6. As these households are those *not* behaving as "permanent-income consumers", this value is in line with conventional estimates such as those in Campbell and Mankiw (1989) or more recently Gali et al. (2007).[11] The discount factor of patient household, β^P $(= 1/R^*)$, is set to 0.99. The instant utility of patient households is $u^P(c) = \ln c$, while that of impatient households is

$$u^I(c) = \begin{cases} \ln c & \text{for } c \leq 1.4 \\ \ln 1.4 + 0.498\,(c - 1.4) & \text{for } c > 1.4, \end{cases}$$

which satisfies the assumptions in Section 2.1, with $\eta = 0.498$ and $c^* = 1.4$. The chosen value of η is equal to the steady state marginal utility of ee households (by far the most numerous amongst the impatient) if they had the same instant utility function as patient households –given the other parameters–, and is meant to minimise differences in asset holding behaviour purely due to differences in instant utility functions.[12] Note that $u^I(c)$ is continuous and (weakly) concave but not differentiable all over $[0, \infty)$; however, it can be made so by 'smooth pasting' the two portions of the function in an arbitrarily small neighbourhood of $c^* = 1.4$. Finally, under our baseline parameterisation and the processes for the exogenous state variables considered in the next Section, it is always the case that $c_t^{eu} < 1.4 < c_t^{ue}$ (as required by our existence conditions.)

The unemployment benefits (δ^I, δ^P) are set to produce a steady state replacement ratio $\delta^j/w^{*j} = 0.4$, for $j = I, P$ (see the discussion in Shimer, 2005, and Section 4.4 for an alternative calibration of this ratio). f^* and s^* are set to their quarter-to-quarter post-war averages (see Appendix C for how quarterly series for f_t and s_t are constructed). By construction, these values produce a steady state unemployment rate $s^*/(f^* + s^*)$ also equal to its post-war average, 5.65%. We set the discount factor of impatient households, β^I, to 0.946. This produces a wealth share $(\Omega n^* a^*)/(n^* k^*)$ of 0.30% for the 60% poorest households, which exactly

[11] In Campbell and Mankiw (1989), a fraction of the households are rule-of-thumb consumers who consume their entire income in every period, possibly due to binding borrowing constraints. However, in Mankiw (2000)'s reinterpretation, "the consumption literature on 'buffer stock saving'can be seen as providing a richer description of this rule-of-thumb behavior. Buffer stock savers are individuals who have high discount rates and often face binding borrowing constraints. Their savings might not be exactly zero: they might hold a small buffer stock as a precaution against very bad income shocks". This is precisely what happens in our model.

[12] More specifically, η solves $\eta = u^{P\prime}(c^{ee*}) = \left(w^{I*} + a^*\left(1 - 1/\beta^P\right)\right)^{-1}$. This equation indicates that the appropriate value of η depends on a^*. Since a^* also depends on η (by (23)), we jointly solve for the fixed point (a^*, η) using a iterative procedure.

matches the corresponding quantile of the distribution of nonhome wealth in the 2007 Survey of Consumer Finances (see Wolff, 2011, Table 2).[13] Finally, the skill premium parameter κ is set to 1.731. Given the other parameters, this value of κ produces a consumption share $C^{I*}/(C^{I*}+C^{P*})$ of 40.62% for the 60% poorest households, which matches the cross-sectional distribution of nondurables in the 2009 Consumer Expenditure Survey.[14] This value of κ is also well in line with the direct evidence on the level of the skill premium, as reported in, e.g., Heathcote et al. (2010) or Acemoglu and Autor (2011).

Our baseline parameterisation is summarised in Table 1. Given those parameters, and in particular the implied low wealth share of impatient households, the existence conditions summarised in Proposition 1 are satisfied by a large margin. In particular, households who fall into unemployment without enjoying full insurance exhaust their buffer stock of wealth within a quarter, and thus live entirely out of unemployment benefits thereafter.

The comparable full-insurance, representative-agent economy is constructed as follows. First, we set $\Omega^{RA} = 0$, so that all households are perfectly insured against unemployment risk. β^P is kept at its baseline value so as to leave the steady state interest factor $1/\beta^P$, and hence capital per effective labour unit k^*, unchanged. The productivity of patient households is set to $\kappa^{RA} = \Omega + (1-\Omega)\,\kappa = 1.24$, so as to maintain the average productivity of the economy unchanged. Since steady state total wealth is $(\Omega + (1-\Omega)\,\kappa)\, n^* k^*$ (see (8)), it takes the same value in the full- and imperfect-insurance economies.

[13]Nonhome wealth is the appropriate wealth concept here, since we are focusing on the part of households' net worth that can readily be converted into cash to provide for current consumption. Besides the net ownership of the primary residence, nonhome wealth excludes consumer durables (whose resale value is low) as well as the social security and pension components of wealth (which cannot be marketed). See Wolff (2010) for a full discussion of this point.

[14]Our empirical couterpart for the consumption share of impatient households is the share of nondurables consumed by the bottom three quintiles of households in terms of pre-tax income, where we define nondurables as in Heathcote et al. (2010).

Parameters	Symb.	Value	*Steady state* (%)	Symb.	Value
Share of impatient hous.	Ω	0.6	Unemployment rate	$1-n^*$	5.65
Disc. factor / patient	β^P	0.99	Wealth share of		
Disc. factor/impatient	β^I	0.946	impatient households	$\frac{\Omega a^*}{k^*}$	0.30
Risk aversion	$\sigma^j, j = I, P$	1	Consumption share of		
Slope/impatient	η	0.498	impatient households	$\frac{C^{I*}}{C^{I*}+C^{P*}}$	40.62
Replacement ratio	δ/w	0.4			
Skill premium parameter	κ	1.731			
Capital share	α	1/3			
Depreciation rate	μ	0.025			
Job-finding rate	f^*	0.789			
Job separation rate	s^*	0.047			

Table 1. Baseline parameters and implied steady state.

4.3 Quantitative implications

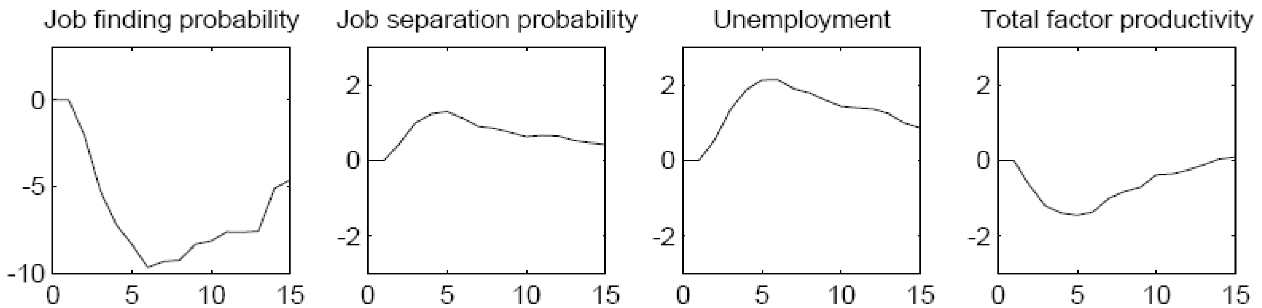
Impulse-response analysis. Our impulse-response analysis focuses on the behaviour of aggregate consumption in a typical recession, which adversely alters the path of labour market transition rates and aggregate income. More specifically, we assume that the economy is at the steady state at date 0. The vector of forcing variables (f_t, s_t, z_t) is subject to a shock at date 2, after which it gradually reverts to the steady state. Our behavioural equations are forward-looking and allow the households to respond to forecasted changes in the forcing terms, notably the transition rates in the labour market (see, e.g., (16)). Since households are likely to anticipate the deterioration of labour market conditions and to adjust their consumption-savings plans accordingly, we allow the joint shock to (f_t, s_t, z_t) to be known one period in advance (that is, at date 1). As with any impulse-response experiment, households have perfect foresight about the evolution of the forcing vector (f_t, s_t, z_t) once the shock is revealed.

The path of the forcing vector after the shock is constructed as follows. We first compute, for each of the eleven US post-war NBER recession, the level-deviations of (f_t, s_t) from their value at the start of the recession, and then take the average of these paths (see Appendix C for details). The first two panels of Figure 2.A shows these average paths, and the third panel the implied path of the unemployment rate. The remaining forcing sequence to be determined is $\{z_t\}$. Given the key role of labour income in determining consumption-saving choices, we select the path of z_t which, given the average trajectories for (f_t, s_t), exactly matches the path of the real wage in the average recession (see Appendix C for a description of the real wage series being used). The implied sequence for z_t is shown in the last panel of Figure 2.A.[15]

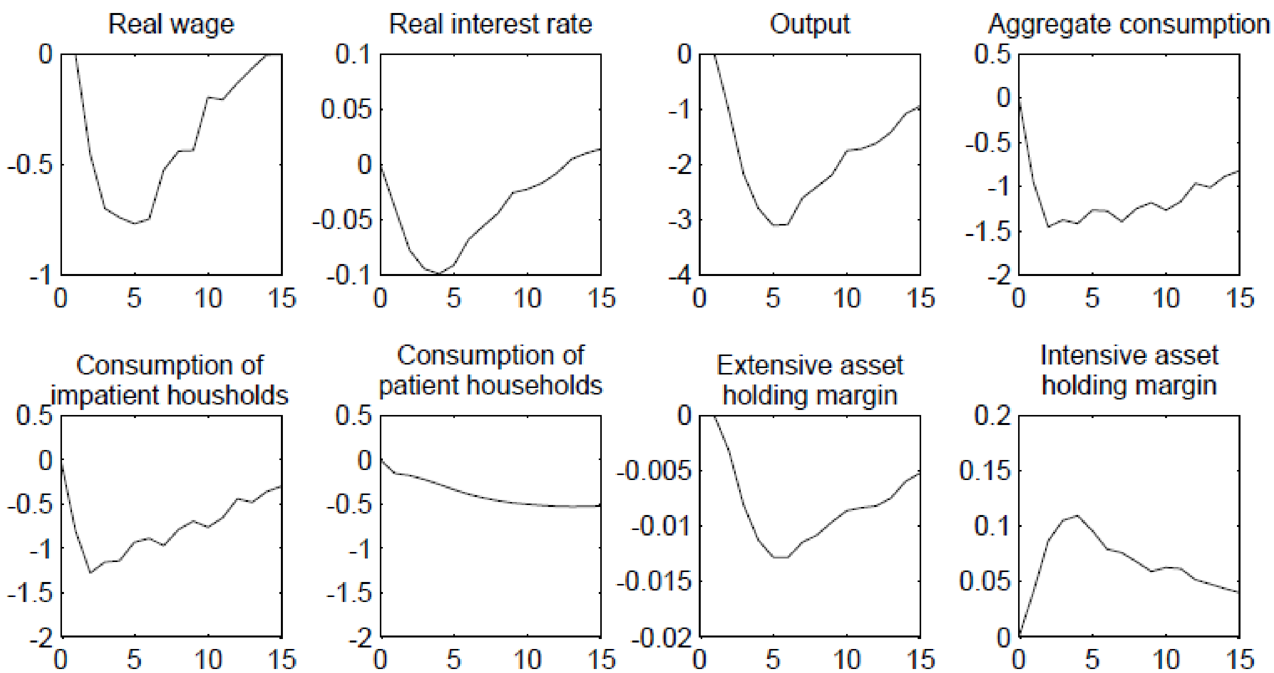
Given our baseline calibration and the path of the forcing vector, we can compute the responses of a number of endogenous variables to a typical recessionary shock. The first row of Figure 2.B. plots the responses of the main aggregate variables, all expressed as percent deviations from the steady state. The first panel plots real wage; by construction, the model exactly reproduces the path of the real wage in the average recession. The second panel shows the adjustment in the (quarterly) real interest rate (which falls from 1%, its value at the steady state, to 0.9% at the trough). The third and fourth panels show the responses of output and aggregate consumption, on which we focus below.

The second row of Figure 2.B. shows how disaggregated variables contribute to movements in the aggregates. The first two panels are the consumption levels of patient and impatient households, C_t^I and C_t^P respectively. We plot $\hat{c}_t^j = (C_t^j - C^{j*})/C^*$, $j = I, P$, implying that the proportional deviations of aggregate consumption from the steady state (first row of Figure 2.B). is $(C_t - C^*)/C^* = \hat{c}_t^P + \hat{c}_t^I$. The consumption of impatient households falls much more abruptly than that of patient households. As discussed above, C_t^I is partly driven by the net savings of impatient households, $\Omega \Delta n_t a_t$ (see equation (18)), which results from the composition of the extensive (Ωn_t) and intensive (a_t) asset holding margins. These variables are plotted in the last two panels of Figure 2.B. One observes that movements along the intensive asset holding margin dominate those along the extensive margin, the overall effect being an increase in precautionary savings and a sharp consumption cut for impatient households.

[15] All our computations in this section are based on the nonlinear model. We have done similar computations with the linearised model, with quantitatively similar results. In the linear case, the ex ante probability that a recession will occur does not affect policy rules, so the outcome of the model is valid for any expected probability of a recession.



A. Paths of forcing sequences and unemployment



B. Paths of endogenous variables

Figure 2. Impulse-response analysis.

Figure 3 compares the response of aggregate consumption implied by our baseline model with i. the actual response of real nondurables and service consumption in the average NBER recession (see Appendix C for details); and ii. that implied by the comparable representative-agent model (i.e., with $\Omega^{RA} = 0$ and $\kappa^{RA} = 1.24$). In the representative-agent economy, aggregate consumption falls mildly on impact (due to the losses in labour income flows that are discounted to the present), and is then fairly smooth; for this reason, the representative-agent model largely misses both the size and the pattern of the consumption trough observed in the data. In contrast, our baseline model produces a much sharper drop in aggregate consumption (due to the the rise in the precautionary motive for impatient households), and one that is of similar magnitude as in the data.

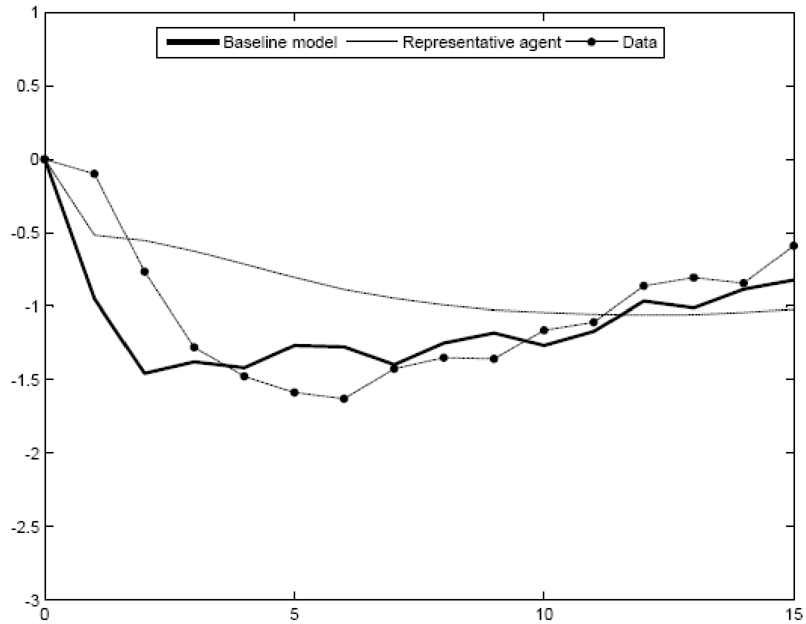


Figure 3. Aggregate consumption

Second-order moments. Our impulse-response analysis above shows our baseline model's response to a shock of large magnitude –a typical NBER recession–, where the role of incomplete markets and precautionary savings are most likely to matter. We now complement our analysis by computing the second-order moments of consumption and income, in the spirit of Krusell and Smith (1998) and much of the business cycle literature. In doing so, we proceed as follows. We first estimate the joint behaviour of the exogenous state vector over the entire postwar period, using a first-order VAR $X_t = AX_{t-1} + \varepsilon_t$, where $X_t = \left[\tilde{f}_t, \tilde{s}_t, \tilde{z}_t\right]'$ includes the HP-filtered job-finding rate, job-separation rate and log-TFP, and where ε_t is the 1×3 vector of innovations. We estimate A and the covariance matrix $\Sigma \equiv \mathrm{Var}\left(\varepsilon_t\right)$ (see Appendix C for the estimated coefficients.) We then run stochastic simulations of the baseline and representative-agent models with a stochastic process for X_t given by the estimated VAR and identified by the matrices A and Σ (the results are very similar whether a linear or a quadratic approximation to these models are used). Hence, households anticipate that the forcing variables follow an auto-correlated process that is consistent with the data.

The top part of Table 2 reports the standard deviation of consumption ($\mathrm{Std}(C_t)$) and its correlation with output ($\mathrm{Corr}(Y_t, C_t)$), such as generated by our baseline model and the representative-agent model. To ease comparison, we also report the size of the consumption trough in a typical recession as implied by the impuse-response experiments described above ($\Delta C/C^*$). Our baseline model produces a consumption volatility and a correlation with output that are both much higher, and also much closer to the data, than the representative-agent model. This result is reminiscent of Krusell and Smith (1998), who also find a similar difference in the consumption-output correlation when moving from their baseline complete-market model to the full-fledged, stochastic-beta version of their heterogenous-agent model. This similarity is instructive given the deep structural differences between the full Krusell and Smith model and our approach. In particular, our framework is based on a simplified representation of households' wealth and consumption heterogeneities. On the other hand, it allows the forcing variables to have continuous support and can be solved via perturbation methods and under rational expectations, which cannot implemented with full-fledged heterogenous-agent models. The two approaches turn out to produce close aggregate time-series properties when subject to a similar stochastic experiment.[16]

Model economies					*Statistics (%)*			
					$\Omega a^*/k^*$	$\Delta C/C^*$	Std(C_t)	Corr(Y_t, C_t)
1.	Data					1.6	.87	.78
2.	Baseline model					1.5	.91	.81
3.	Representative agent					1.1	.54	.62
	Ω	δ/w^*	κ	β^I				
4.	0.3	.4	1.7	.946	.23	1.1	.62	.81
5.	0.6	.955	1.7	.946	.00	1.6	.68	.94
6.	0.6	.4	1.0	.946	.38	1.8	1.1	.79
7.	0.6	.4	1.7	.960	.61	1.4	.87	.83

Table 2. Summary statistics.

Sensitivity. The lower part of Table 2 reports the sensitivity of our results with respects to changes in the share of impatient household (Ω, Model 4), the replacement ratio (δ^j/w^{*j}, $j = I, P$, Model 5), the skill premium parameter (κ, Model 6) and the subjective discount factor for impatient households (β^I, Model 7). For each model, we report the same statistics as before, as well as the implied steady state wealth share of impatient households ($\Omega a^*/k^*$).

[16] It is of course possible to simulate the *deterministic* response of a full-fledged heterogenous-agent model to a once-occuring shock to the forcing variables, as we did in Section 4.3 for our baseline model. However, we find the shape of impulse-response functions to be highly sensitive to the parameters that determine the cross-sectional distribution of wealth (e.g., the stochastic processes for the discount factor and labor income processes, the severity of the borrowing constraint, etc.)

Models 4 and 5 correspond to economies which are closer to the representative-agent model than our baseline specification, and hence where market incompleteness plays a more limited role in driving the response of consumption to aggregate shocks. In Model 4, the share of impatient households is reduced; perhaps unsurprisingly, this models predicts a mild response of aggregate consumption in a recession (comparable to that in the representative-agent economy) and a low unconditional variability of this variable relative to the data. In Model 5, the number of impatient households is maintained at its baseline value, but these households are endowed with better insurance opportunities; namely, the steady state replacement ratio is set to the high value 0.955, while social contributions are raised to satisfy the balanced-budget condition (10). This economy is one where the cost associated with falling into unemployment is low, as suggested by Hadgedorn and Manovskii (2008). In this situation, the incentive to hoard assets for precautionary purposes vanishes and the condition on the strict positivity of asset holdings by employed, impatient households in (12) is violated. Since the borrowing constraint becomes binding for all impatient households (including the employed), they collectively behave like pure liquidity-constrained "rule-of-thumb" consumers. This model economy matches the size of the consumption trough in a recession; however, by tightly linking aggregate consumption to aggregate income, it greatly overestimates the unconditional correlation between these two variables.

In Model 6, the heterogeneity in labour efficiency across patient and impatient households that was assumed in the baseline specification is removed. Since all households earn the same wage, wealth is slightly more equally distributed. More importantly, this specification overestimate the consumption of the 60% poorest in the population, as those end up consuming 53% total consumption (not reported in Table 2), against 41% in the data (see Table 1). Since the consumption of impatient households responds more to aggregate shocks, the composition effect leads to an overestimation of both the size of the consumption trough in a recession and the unconditional volatility of consumption. Finally, Model 7 is one in which impatient households are more patient than in the baseline specification, and consequently hold a larger fraction of total wealth. The results are close to those under the baseline specification, except perhaps for the somewhat lower size of the consumption trough (due to the fact that impatient households are better self-insured in the first place).

5 Concluding remarks

In this paper, we have proposed a tractable general equilibrium model of households' behaviour under incomplete insurance and time-varying precautionary savings, and then gauged its ability to shed light on the dynamics of aggregate consumption over the business cycle. In contrast to earlier attempts at constructing tractable versions of models with heterogenous agents, the specificity of ours has been to combine i. a realistic representation of households' labour income risk –modelled as resulting from the combined effects of persistent changes in the equilibrium real wage and in the probabilities to transit across employment statuses; and ii. the reduction of the model's dynamic to a small-scale system solved under rational expectations, thanks to exact cross-household aggregation. While the present contribution has deliberately focused on a simple model specification (by drastically limiting the number of household types) and quantified its most direct implication (the impact of aggregate shocks on aggregate consumption), it should be clear that the very same framework can be used to incorporate, and study the implications of, incomplete markets and borrowing constraints in much richer models. To be more precise, we see two contexts in which this framework may be useful. First, it can be used to interact incomplete insurance with other frictions that are commonly thought of as important for understanding business cycles but remain difficult to incorporate into full-fledged heterogenous-agent models, such as nominal rigidities.[17] Second, since our approach allows all variables (including the exogenous state vector) to have continuous support, it is directly amenable to structural estimation –provided that the model is sufficiently enriched to improve its empirical fit.

Appendix

A. Endogenous transitions rates

This appendix shows how (f_t, s_t) can be determined by the job creation policy of the firm in a labour market with search and matching frictions. We use the same timing convention and employment contract as in Hall (2009), which allows us to be as close as possible to our baseline model with exogenous transition rates. Moreover, for expositional conciseness we assume that $\kappa = 1$ (i.e., all households are equally productive), but the argument directly extends to the case where $\kappa \neq 1$.

[17] As far as we are aware, the only heterogenous-agent model that incorporates nominal frictions is that of Guerrieri and Lorenzoni (2011), who assume constant nominal prices.

More specifically, our timing is as follows. At the very beginning of date t, a fraction ρ of existing employment relationships are broken, thereby creating a job seekers' pool of size $1-(1-\rho)\,n_{t-1}$ (that is, unemployment at the end of date $t-1$, $1-n_{t-1}$, *plus* the broken relationships at the beginning of date t, ρn_{t-1}). Members of this pool then have a probability f_t to find a job within the same period, and stay unemployed until the end of the period with complementary probability. It follows that the period-to-period separation rate is $s_t=\rho\,(1-f_t)$.

The job-finding rate, f_t, is determined as follows. Given its knowledge of $1-(1-\rho)\,n_{t-1}$ and z_t, the representative firm posts v_t vacancies at cost $c>0$ each and a fraction λ_t of which are filled in the current period. Total employment at the end of date t is thus:

$$n_t=(1-\rho)\,n_{t-1}+\lambda_t v_t. \tag{28}$$

The vacancy filling rate λ_t is related to the vacancy opening policy of the firm via the matching technology. The number of matches M_t formed at date t is assumed to depend on both the size of the job seekers' pool and the number of posted vacancies, v_t, according to the function $M_t=M\,(1-(1-\rho)\,n_{t-1},v_t)$, which is increasing and strictly concave in both arguments and has CRS. Thus, the vacancy-filling rate satisfies $\lambda_t=M_t/v_t=m\,(\theta_t)$, where $\theta_t\equiv v_t/\,(1-(1-\rho)\,n_{t-1})$ is the market tightness ratio, and where the function $m\,(\theta_t)\equiv M\left(\theta_t^{-1},1\right)$ is strictly decreasing in θ_t.

Once matched, the households and the firm split the match surplus according to bilaterally efficient dynamic contracts that are negotiated at the time of the match and implemented as planned for the duration of the match. Following Hall (2009) and Stevens (2004), we restrict our attention to a simple class of dynamic contracts whereby the firm pays the worker its full marginal product, except at the time of the match when the worker is paid below marginal product. The profit flow extracted by the firm on new matches motivates –and finances– the payment of vacancy opening costs, but existing matches generate no quasi-rents thereafter. Formally, this arrangement is equivalent to a "fee contract" in which any matched worker i enjoys the wage $w_t=z_tG_2\,(K_t,n_t)$ at any point in time but pays a fixed fee $\psi_t>0$ to the firm at the time of hiring; that is, the worker is actually paid $\tilde{w}_t=w_t-\psi_t>0$ during the (one-period) probation time and the full wage thereafter. We let ψ_t respond to the aggregate state, i.e., $\psi_t=\psi\,(z_t)$, $\psi'\,(.)>0$. The representative firm maximises its instantaneous profit flow

$$\Pi_t=z_tG\,(K_t,n_t)-n_tw_t-(R_t-1+\mu)\,K_t+v_t\,(\lambda_t\psi_t-c)\,, \tag{29}$$

subject to (28), and taking n_{t-1}, λ_t, R_t, as well as the contract $(\tilde{w}_t,w_t)$, as given. The optimal choice of capital per employee is given by equation (7). On the other hand, from (29) the firm expands vacancy openings until $z_t G_2(K_t, n_t)\lambda_t - \lambda_t w_t + \lambda_t \psi_t - c = 0$. Since $z_t G_2(K_t, n_t) = w_t$ in the class of contracts under consideration, the economywide vacancy-filling rate that results from these openings is:

$$\lambda_t = c/\psi(z_t) \equiv \lambda(z_t), \ \lambda'(.) < 0. \tag{30}$$

From (29)–(30) and the CRS assumption, the firm makes no pure profits in equilibrium (i.e., old matches generate no profit, while the quasi-rent extracted from new matches is exhausted in the payment of vacancies costs). From the matching function specified above, the tightness ratio that results from the optimal vacancy policy of the firm is $\theta_t = m^{-1}(\lambda_t)$. Hence, the job-finding rate in this economy is:

$$f_t = \lambda_t . m^{-1}(\lambda_t) \equiv f(z_t), \ f'(.) > 0, \tag{31}$$

and the job separation rate is $s(z_t) = \rho(1 - f(z_t))$. Note that under this structure the firm's problem is static and thus unambiguous despite the fact that impatient and patient households do not share the same pricing kernel.

B. Equilibria with many wealth states

In this Appendix, we use the same constructive approach as in the body of the paper in order to derive a set of necessary and sufficient conditions for the existence and uniqueness of equilibria with many wealth states. We derive these conditions at the steady state (which as before satisfies $R^* = 1/\beta^P$, (22) and (24)), knowing that those will still hold in the vicinity of the steady state provided that aggregate shocks are not too large. More specifically, we generalise our approach to construct equilibria having the following properties: i) given a period utility function for impatient households with the shape depicted in Figure 1, i) all unemployed household consume less than c^*; ii) all employed households consume more than c^*; iii) none of the employed face a binding borrowing constraint; and iv) unemployed households do face a binding borrowing constraint after $m < \infty$ consecutive periods of unemployment (that is, asset liquidation takes place gradually rather than in one period, but is achieved in finite time).

Let us call 'k-unemployed households' those who are unemployed for exactly $k \geq 1$ consecutive periods at date t (and thus who were still employed at date $t - k$). For example, 1-unemployed households are unemployed at date t and but were employed at date $t-1$. Similarly, we denote by 'k-employed households' those employed at date t who where unemployed from date $t-k-1$ to date $t-1$. By extension, 0-employed households are those employed date t who were also employed at date $t-1$.

Conjecture that i) all k-unemployed households consume the same amount c_k^u and hold the same asset wealth, denoted a_k; ii) all employment households hold the same asset wealth, denoted a_0; and iii) all k-employed households consume the same amount c_k^e. We will show that it is indeed the case in equilibrium. By assumption, the period utility function has the following property:

$$u(c) = \begin{cases} \tilde{u}(c) & \text{if } c \leq c^* \\ \tilde{u}(c^*) + \eta(c - c^*) & \text{if } c > c^* \end{cases} \tag{32}$$

Moreover we assume that $\tilde{u}$ is differentiable and strictly concave, with $\tilde{u}'(c) > \eta$ for all $c < c^*$ and $\lim_{c\to 0^+} \tilde{u}'(c) = +\infty$. Finally, we assume that:

$$\eta < \beta^I R^* \left[(1-s^*)\eta + s^* \tilde{u}'\left(\delta^I\right)\right] \tag{33}$$

We will first characterize the equilibrium wealth distribution when (33) holds. Then we will show that when it does not, then all impatient households face a binding borrowing limit and hence simply behave like 'rule-of-thumb' consumers.

Consumption and asset levels. Under our conjectured equilibrium, the Euler equations determining the distribution of wealth are:

$$\eta = \beta^I \left[(1-s^*)\eta + s^* \tilde{u}'(c_1^u)\right] R^*, \tag{34}$$

$$\tilde{u}'(c_k^u) = \beta^I \left[f^* \eta + (1-f^*) \tilde{u}'\left(c_{k+1}^u\right)\right] R^* \text{ if } m \geq 2 \text{ and for } k = 1...m-1, \tag{35}$$

where (34) is the Euler equation for employed households and (35) are the Euler equations for unemployed households before the borrowing constraint starts binding. From the households' budget constraints, the consumption levels of unemployed households are

$$c_k^u = \delta^I + a_{k-1}R^* - a_k \text{ if } m \geq 2 \text{ and for } k = 1...m-1, \tag{36}$$

$$c_m^u = \delta^I + a_{m-1}R^*, \tag{37}$$

$$c_k^u = \delta^I \qquad \text{for } k > m, \tag{38}$$

while those of employed households are

$$c_j^e = w^*(1-\tau^*) + a_j R^* - a_0 \text{ for } j = 0, ...m \tag{39}$$

The conditions for the existence of an equilibrium with $m+1$ wealth states can be stated as follows:

$$c_k^e > c^* > c_j^u \text{ for all } k \geq 0 \text{ and } j \geq 1, \tag{40}$$

$$a_k > 0 \text{ for } k \geq 0, \tag{41}$$

$$\tilde{u}'\left(c_m^u\right) > \beta\left[f^*\eta + \left(1-f^*\right)\tilde{u}'\left(\delta\right)\right] R^*. \tag{42}$$

Inequality (40) ensures that the ranking of consumption levels is consistent with the assumed instant utility function. Inequality (41) states that asset holdings are positive before the constraint binds. Equation (42) states that the borrowing constraint is binding after m periods of unemployment. We now derive a set of necessary and sufficient conditions for equations (34)–(42) to hold, thereby implying the existence of an equilibrium with limited cross-sectional heterogeneity. Moreover, we will show that when this equilibrium exist, then it is unique.

Existence and uniqueness of the equilibrium. To establish existence and uniqueness, we first construct a function $F\left(m, a_{m-1}\right)$ that depends on the number of liquidation periods for unemployed households before the constraint binds (m) and the last strictly positive wealth level before the constraint binds (a_{m-1}). If the borrowing constraints binds after m consecutive periods of unemployment, then we know that $a_m = 0$. In words, for every possible m, F is defined over all possible values of a_{m-1}. The goal is to use F to show that both m and a_{m-1} are unique given the deep parameters of the model. To construct F, note that from (35) and (36) we have

$$\tilde{u}'\left(c_{m-1}^u\right) = \beta^I\left[f^*\eta + \left(1-f^*\right)\tilde{u}'\left(\delta^I + a_{m-1}R^*\right)\right] R^*.$$

Using (35) $m-2$ times to eliminate c_{m-1}^u, c_{m-2}^u and c_2^u, and then (34) to eliminate c_1^u , we obtain:

$$\begin{aligned}\left[\beta^I R^*\left(1-s^*\right)-1\right]\eta + s^* f^*\eta\left(\beta^I R^*\right)^2\left(\sum_{k=0}^{m-2}\left(\beta^I R^*\left(1-f^*\right)\right)^k\right)& \\ +\left(\beta^I R^*\right)\left(\beta^I R^*\left(1-f^*\right)\right)^{m-1} s^* u'\left(\delta^I + a_{m-1}^u R^*\right) = 0&\end{aligned} \tag{43}$$

The previous equality is valid for $m \geq 2$. If $m = 1$, one can directly use (34) to find a_0, which corresponds to the baseline case analysed in the body of the paper.

Lemma 1 *Define the function* $F : \mathbb{N}\times]-\delta/R^*;+\infty[\rightarrow \mathbb{R}$*, such that*

$$F\left(1, x\right) \equiv \beta^I R^* s^* \tilde{u}'\left(\delta^I + xR^*\right) - \left[1-\beta^I R^*\left(1-s^*\right)\right]\eta$$

and, for $m \geq 2$,

$$F(m,x) \equiv \frac{s^* f^* \eta \left(\beta^I R^*\right)^2}{1-\beta^I R^*(1-f^*)} - \left[1-\beta^I R^*(1-s^*)\right]\eta \qquad (44)$$
$$+ s^* \beta^I R^* \left[\tilde{u}'(\delta + xR^*) - \frac{f^* \eta \beta^I R^*}{1-\beta^I R^*(1-f^*)}\right]\left(\beta^I R^*(1-f^*)\right)^{m-1}.$$

Then, for any $m \geq 1$, a_{m-1} satisfies $F(m, a_{m-1}) = 0$.

The function F is obtained by factorising the left hand side of (43), and it only depends on the deep parameters of the model. The following lemma uncovers some useful properties of F.

Lemma 2 *If (33) holds then*
i) F is strictly decreasing in both argument
ii) For all $m \geq 1$, there exists a unique x_{m-1} such that $F(m, x_{m-1}) = 0$. Moreover, the series x_{m-1} is strictly decreasing in m and we have $x_0 > 0$ and $\lim_{m\to\infty} x_{m-1} = -\delta^I/R^*$.

Proof. i) We first note that $\tilde{u}'\left(\delta^I + xR^*\right) - \frac{\beta^I R^* f^* \eta}{(1-\beta^I R^*(1-f^*))} > 0$, since $1 > \beta^I R^*$ and $\tilde{u}'(.) \geq \eta$. Using this inequality, the expression for $F(1,x)$ above and equation (44) with $m = 2$, we have $F(1,x) > F(2,x)$. Moreover, for all $m \geq 2$ and $x > -\delta^I/R^*$ we have $F(m-1,x) > F(m,x)$, that is, $F(m,x)$ is strictly decreasing in m. Finally, $F(m,x)$ is decreasing in x for all m because $\tilde{u}' < 0$. ii) Note first that $\lim_{x\to-\delta^I/R^*} F(m,x) = +\infty$ for all $m \geq 1$. Moreover, (33) implies that $F(1,0) = \beta^I R^* s^* \tilde{u}'\left(\delta^I\right) - \left[1-\beta^I R^*(1-s^*)\right]\eta < 0$. Since $F(1,x)$ is continuous and monotonic in x, by the theorem of continuous functions there is an x_0 such that $F(1,x_0) = 0$. Now, since $F(m,x_0)$ is decreasing in m we have $F(2,x_0) < 0$; and, since $F(2,x)$ is continuous and monotonic in x, there exists a unique x_1 such that $F(2,x_1) = 0$. Then, by induction it is easy to show that $F(m, x_{m-2}) < 0$, and that, for all $m \geq 1$, there exists a unique x_{m-1} such that $F(m, x_{m-1}) = 0$. Since $F(m,x)$ is strictly decreasing in both argument, the series x_{m-1} defined by $F(m, x_{m-1}) = 0$ is decreasing in m. Finally, using (44), one finds that $F(m, x_{m-1}) = 0$ is equivalent to

$$\tilde{u}'\left(\delta^I + x_{m-1}R^*\right) = \frac{1}{s^*\beta R^*}\left([1-\beta R^*(1-s^*)]\eta - \frac{s^* f^* \eta (\beta R^*)^2}{1-\beta R^*(1-f^*)}\right)\frac{1}{(\beta R^*(1-f^*))^{m-1}}$$
$$+ \frac{f^*\eta\beta R^*}{1-\beta R^*(1-f^*)} \qquad (45)$$

The right hand side of this equation goes to $+\infty$ as m goes to $+\infty$. Since $\lim_{c\to 0^+} \tilde{u}'(c) = +\infty$, this implies that $x_{m-1} \to -\delta^I/R^*$ as $m \to +\infty$. ■

Using the properties of F, we may prove the uniqueness of the equilibrium, provided that the latter exists.

Proposition 2. If there exists an equilibrium with limited heterogeneity, then it is unique. Moreover, borrowing constraints bind after a finite number of consecutive periods of unemployment.

Proof. Define

$$m^* = \max\{m | x_{m-1} \geq 0\}, \tag{46}$$

that is, m^* is the largest m for which x_{m-1} is nonnegative. Since x_{m-1} is strictly decreasing, $x_0 > 0$ and $\lim_{m \to +\infty} x_{m-1} = -\delta^I / R^*$, m^* is finite and uniquely defined. We now show (by contradiction) that if there exists an equilibrium with limited heterogeneity, then the borrowing constraint must be binding after exactly m consecutive periods of unemployment. Suppose that there is also an equilibrium in which households face binding borrowing constraints after $n \neq m^*$ consecutive periods of unemployment. Then n would satisfy $F(n, a_{n-1}) = 0$. First, it is impossible that $n > m^*$, since in this case we would have $a_{n-1} < 0$, a contradiction. Second, if we could find $n < m^*$ then we would have $F(m^*, a_{m^*-1}) = F(n, a_{n-1}) (= 0)$. Rearranging this equality, it can be shown that it would imply that if the borrowing constraint binds after n consecutive periods of unemployment then $a_{m-1} < 0$, a contradiction again. Thus, the only possible equilibrium is such that $m = m^*$, where m^* is finite. ■

We now characterize the equilibrium allocation with m^* liquidation periods.

Lemma 3 *In an equilibrium with limited heterogeneity, i) c_k^u is strictly decreasing in $k = 1...m^*$; and ii) a_j is strictly decreasing in $j = 0...m^* - 1$*

Proof. i) is proven by induction. From (35), if $\tilde{u}'(c_k^u) > \eta f / \left(1/\beta^I R^* - 1 + f\right)$, then $\tilde{u}'\left(c_{k+1}^u\right) > \tilde{u}'(c_k^u)$ and hence $c_{k+1}^u < c_k^u$. Now, from (34) we have

$$\tilde{u}'(c_1^u) = \left(1 + \frac{1/(\beta R^*) - 1}{s^*}\right) \eta.$$

Since $\beta^I R^* < 1$, we know that $\tilde{u}'(c_1) > \eta$; and since $f / \left(1/\beta^I R^* - 1 + f\right) < 1$, it implies that $\tilde{u}'(c_1) > \eta f / \left(1/\beta^I R^* - 1 + f\right)$. ii) For $m^* = 1$, then from (33) we have $a_0 > 0$. For $m^* = 2$, since $c_{m^*-1}^u = \delta^I + a_{m^*-2} R^* - a_{m^*-1} > c_m^u = \delta^I + a_{m^*-1} R^*$ we have $a_{m^*-2} > a_{m^*-1}(1 + 1/R^*) > a_{m^*-1}$. For $m \geq 3$, we again reason by induction. First, $a_{m^*-2} > a_{m^*-1}$, for the same reason as when $m^* = 2$. Second, if $a_{m^*-2} > a_{m^*-1}$ then $a_{m^*-3} > a_{m^*-2}$; Indeed, from (36)-(38) and the fact that $c_{k-1}^u > c_k^u$, $k = 0, ...m^* - 1$ we have $a_{m^*-3} - a_{m^*-2} > (a_{m^*-2} - a_{m^*-1}) / R^*$. ■

We may then construct the states of the wealth distribution as follows. Given m^* in (46), the sequence $\{a_{m^*-i}\}_{i=0}^{m^*}$ is given by $a_{m^*}=0$, a_{m^*-1} that solves $F(m^*, a_{m^*-1})=0$, and then the recursion

$$\begin{aligned} &\tilde{u}'\left(\delta^I + R^* a_{k-2} - a_{k-1}\right) \\ &= \beta^I\left[f^*\eta + (1-f^*)\,\tilde{u}'\left(\delta^I + R^* a_{k-1} - a_k\right)\right] R^* \text{ if } m^* \geq 2 \text{ and for } 2 \leq k \leq m^* \end{aligned} \tag{47}$$

until a_0. This recursion also allows us to find a_0 as a function of the deep parameters of the model. Indeed, define the function G and net dis-saving X_k as follows:

$$G(X) = \tilde{u}'^{-1}\left(\beta^I R^*\left[f^*\eta + (1-f^*)\,\tilde{u}'\left(\delta^I + X\right)\right]\right) - \delta^I, \tag{48}$$

$$X_k \equiv R^* a_{k-1} - a_k. \tag{49}$$

Then, (47) can be written as $X_{k-1} = G(X_k)$ for $k = 2..m^*$. Iterating this equation gives:

$$X_k = G^{(m^*-k)}\left(R^* a_{m^*-1}\right) \text{ for } k = 1...m^* - 1,$$

where $G^{(i)}$ is the i-th iteration of G. The a_ks are then recovered from (49). In particular,

$$a_0 = \sum_{j=1}^{m^*-1} \frac{G^{(j)}\left(R^* a_{m^*-1}\right)}{(R^*)^j} + \frac{a_{m^*-1}}{(R^*)^{m^*-1}},\ a_1 = \sum_{j=1}^{m^*-2} \frac{G^{(j)}\left(R^* a_{m^*-1}\right)}{(R^*)^j} + \frac{a_{m^*-1}}{(R^*)^{m^*-2}} \tag{50}$$

We can now prove the following proposition.

Proposition 3. If $w^*(1-\tau^*) - a_0 > c^* > \delta + R^* a_0 - a_1$, where a_0 and a_1 are given (50). Then, the equilibrium with reduced heterogeneity exists and is unique.

Proof. We have shown that if (33) holds, then there is a unique m^* and a unique sequence c_k^u, $k = 1...m$ such that the Euler equations (34)–(35) hold and the borrowing constraint is binding after m continuous periods of unemployment. For this allocation to be an equilibrium with limited heterogeneity, one has to show that the ranking of consumption levels (40) is satisfied. Since both c_k^u and a_k are decreasing in k, c_j^e in (39) is decreasing in j. Hence, a sufficient condition for the postulated ranking of consumption levels to hold is $c_m^e > c^* > c_1^u$, which is equivalent to $w^*(1-\tau^*) - a_0 > c^* > \delta + R^* a_0 - a_1$. If this last condition hold, then the allocation $\{c_k^u, a_k\}_k$ is the unique equilibrium. ■

Construction of the period utility function. Taking a utility function $\tilde{u}$ and a coefficient η, we can now state the conditions under which one can construct an equilibrium with limited cross-sectional heterogeneity.

Proposition 4. For a given function $\tilde{u}$, differentiable, increasing and strictly concave and with $\lim_{c\to 0^+} \tilde{u}'(c) = +\infty$ and a given $\eta > 0$, one can construct an increasing and concave utility function u such that a limited heterogeneity equilibrium exists provided that
i) $w^*(1-\tau^*) - a_0 > \delta + R^* a_0 - a_1$,
ii) $\tilde{u}'(\delta + R^* a_0 - a_1) \geq \eta$,
where a_0 and a_1 are given by (50).

Indeed, if i) holds, then one can finds a c^* such that $w^*(1-\tau^*) - a_0 > c^* > \delta + R^* a_0 - a_1$. One may then construct a utility function u differentiable all over $[0, \infty)$ by smooth-pasting $\tilde{u}(.)$ with the linear part of $u(.)$ in an arbitrarily small neighborhood of c^*. The function u can be (weakly) concave provided that ii) holds.

C. Data

In Section 4, we use quarterly time series for labour market transition rates, aggregate consumption, the real wage, and total factor productivity over the period 1948Q1-2011Q1.

Labour market transition rates. For the job-finding and job-separation probabilities, we proceed as follows. First, we compute monthly job-finding probabilities using CPS data on unemployment and short-run unemployment, using the two-state approach of Shimer (2005, 2007). (As suggested by Shimer, 2007, the short-run unemployment series is made homogenous over the entire sample by multiplying the raw series by 1.1 from 1994M1 onwards). We then compute monthly separation probabilities as residuals from a monthly flow equation similar to (1). Using these two series, we construct transition matrices across employment statuses for every month in the sample, and then multiply those matrices over the three consecutive months of each quarter to obtain quarterly transition rates (this implies that cyclical fluctuations in the quarter-to-quarter separation rate partly reflect the changes in the job-finding rate that occur at monthly frequency).

Our impulse-response experiment uses as inputs the movements in these transition rates that take place over a typical recession. For each of the eleven US post-war NBER recession, we compute the level-deviation of both rates from their value in the first quarter of the recession, and for the following 15 quarters. We then take the arithmetic average of these paths. These are represented in the first two panels of Figure 2.A. The stochastic simulations that we run in order to compute the model's second-order moments requires the estimation of VAR that includes the cyclical component of labour market transition rates. The series we use is an HP-filtered version of the quarterly transition rates (with smoothing parameter 1600).

Real wage. Our quarterly aggregate real wage series is constructed as follows:

$$\text{Real wage} = \frac{\text{Total wages and salary accruals (a)} + \frac{2}{3} \times \text{proprietors' income (b)}}{\text{Civilian employment (c)} \times \text{PCE price index (d)}},$$

where (a), (b) and (d) are taken from the NIPA tables and (c) is from the CPS. The real wage series is not stationary, but we extract its cyclical component, expressed as proportional deviations from the trend, by HP-filtering the log of the series with smoothing parameter 1600. We then construct movements in the real wage over the average recession in the same way as with labour market transition rates: first, for each post-war recession, we compute the deviation of the cyclical component of the wage from its value at the onset of the recession and over the following 15 quarters; and second, we take the average of these eleven realisations. As explained in section 4.3, in our impulse-response experiment the path of the technology parameter z_t is reversed-engineered so that the baseline model exactly produces the observed real wage series. Hence, the latter appear as an 'endogenous' variable in the first panel of Figure 2.B.

Aggregate consumption. Our quarterly real consumption series is obtained by dividing nominal spending on nondurable goods and services by the PCE price index (all from the NIPA tables). The cycle component of the series is extracted by HP-filtering the log of the series (with smoothing parameter 1600). The 'data' line in Figure 3 shows the dynamics of aggregate consumption in an average recession, which again is obtained by averaging the path of aggregate consumption over the eleven post-war recessions in the US Note that in the models' timing, the forcing vector is known one period in advance, so consumption (and assets) moves one period before the other variables. To be consistent with this timing, when constructing the path of aggregate consumption in each recession we normalise consumption to zero one quarter before the start of each recession.

Total factor productivity. Our series for total factor productivity is computed as in Ríos-Rull and Santaeulàlia-Llopis (2010). The estimated coefficient and variance-covariance matrices used in Section 4.2 are

$$A = \begin{bmatrix} 0.573 & 0.501 & -0.096 \\ -0.111 & 0.487 & 0.023 \\ -0.680 & -2.474 & 0.877 \end{bmatrix}, \ \Sigma = \begin{bmatrix} 3.16 \times 10^{-5} & 1.65 \times 10^{-6} & -4.11 \times 10^{-6} \\ 1.65 \times 10^{-6} & 2.59 \times 10^{-4} & -3.20 \times 10^{-5} \\ -4.11 \times 10^{-6} & -3.20 \times 10^{-5} & 9.14 \times 10^{-6} \end{bmatrix}.$$

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