Dynamic Adverse Selection: A Theory of Illiquidity, Fire Sales, and Flight to Quality*

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Abstract

We develop a theory of equilibrium in asset markets with adverse selection. Traders can buy and sell an asset at any price. Sellers recognize that their trades may be rationed if they ask for a high price, while buyers recognize that they can only get a high quality good by paying a high price. These beliefs are consistent with rational behavior by all traders. In the resulting equilibrium, the existence of low-quality assets reduces the liquidity and price-dividend ratio in the market for high quality assets. The emergence or worsening of an adverse selection causes a fire sale, with the price and liquidity of all such assets declining. The price of other assets that do not suffer from adverse selection may rise, a flight to quality. If a large player purchases and destroys all the low quality assets, the liquidity and price-dividend ratio will increase for high quality assets.

1 Introduction

This paper develops a dynamic equilibrium model of asset markets with adverse selection. Sellers can attempt to sell a durable asset at any price. Buyers must form rational expectations about the type of asset that is available at each price. In equilibrium, sellers are rationed by a shortage of buyers at all prices except the lowest one, and it is increasingly difficult to sell an asset at higher prices. This keeps the owners of low quality assets from trying to sell them at high prices. On the other hand, the owners of high quality assets are willing to set a high price despite the low sale probability because the asset is worth more to them if they fail to sell it.

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Our model offers an abstract view of an illiquid asset market, for example the market for asset-backed securities during the 2007–2008 financial crisis. Prior to the crisis, market participants viewed AAA securities as a safe investment, indistinguishable from Treasuries; indeed, they were treated as such by banking regulators. In the early stages of the crisis, investors started to recognize that some of these securities were likely to pay less than face value. Moreover, it was difficult to determine the exact assets that backed each individual security. Anticipating that she might later have to sell it, at this point it started to pay for the owner of an asset to learn its quality. On the other hand, it did not pay potential buyers to investigate the quality of all possible assets because they did not know which assets would later be for sale. This created an adverse selection problem, where sellers have superior information than buyers, as in the classic market for lemons (Akerlof, 1970).

We predict that within an asset class, such as AAA-rated mortgage backed securities, a seller should always be able to sell an asset at a sufficiently low price. However, the owners of good quality assets will choose to hold out for a higher price, recognizing that there will be a shortage of buyers at that price and so it will take time to sell the asset. Moreover, the price that buyers are willing to pay for a high quality asset will be depressed because the market is less liquid. That is, even if a buyer somehow understood that a particular mortgage-backed security would pay the promised dividends with certainty, he would pay less for it than for a Treasury because he would anticipate having trouble reselling the MBS to future buyers who don't have his information. Illiquidity therefore serves to further depress asset prices. In particular, the ability of sellers to learn the quality of their assets will depress the liquidity and may depress the value of all securities even if the average quality is unchanged. We view an event where sellers start to learn the quality of the assets in their portfolio as a fire sale.[1] On the other hand, buyers still would like to reinvest their income in some asset, and so the decline in the demand for MBS will boost the demand for other assets that do not suffer from an adverse selection problem, such as Treasury bonds. Thus our model generates a flight-to-quality.

An obvious solution to this problem is to have a third party evaluate the quality of the assets. Indeed, this is the role that the rating agencies were supposed to play. But the rating agencies lost their credibility during the crisis and there was no one with the reputation and capability to take their place. We find instead that there may have been a role for an investor

[1]For a detailed description of the first phase of the crisis and an analysis of the source of the adverse selection problem, see Gorton (2008). This view of the crisis is consistent with Dang, Gorton and Holmström (2009), who conclude, "Systemic crises concern debt. The crisis that can occur with debt is due to the fact that the debt is not riskless. A bad enough shock can cause information insensitive debt to become information sensitive, make the production of private information profitable, and trigger adverse selection. Instead of trading at the new and lower expected value of the debt given the shock, agents trade much less than they could or even not at all. There is a collapse in trade. The onset of adverse selection is the crisis."

with deep pockets, such as a government, to purchase low quality assets and alleviate the illiquidity of high quality ones. In particular, suppose the government stood ready to buy all assets at a moderate price. Any asset which the seller believed was worth less than that price, even if fully liquid, would be sold to the government, which in turn would take a loss on its purchases. The elimination of trade in low quality assets moderates the adverse selection problem. This makes all other assets more liquid and more expensive. Thus asset purchases can potentially alleviate both illiquidity and insolvency.

Our model is deliberately stylized. Assets are perfectly durable and pay a constant dividend, a perishable consumption good. Better quality assets pay a higher dividend. Individuals are risk-neutral and have a discount factor that shifts randomly over time, creating a reason for trade. The only permissible trades are between the consumption good and the asset. Still, we believe this framework is useful for capturing our main idea that illiquidity may serve to separate high and low quality assets. In particular, it is a dynamic general equilibrium model in which the distribution of asset holdings evolves endogenously over time as individuals trade and experience preference shocks. We define a competitive equilibrium in this environment and prove that it is unique. In equilibrium, higher quality assets trade at a higher price but with a lower probability. The expected revenue from selling an asset, the product of its price and trading probability, is decreasing in the quality of the asset.

We also show that the trading frictions in this environment do not depend on any assumptions about the frequency of trading opportunities. Even with continuous trading opportunities, there are not enough buyers in the market for high quality assets and so it takes a real amount of calendar time to sell at a high price. This is in contrast to models that emphasize illiquidity in asset markets due to search frictions, such as Duffie, Gârleanu and Pedersen (2005), Weill (2008), and Lagos and Rocheteau (2009), where the economy converges to the frictionless outcome when the time between trading opportunities goes to zero. In our adverse selection economy, real trading delays are essential for separating the good assets from the bad ones. Of course, in reality adverse selection and search frictions may coexist in a market, and it is indeed straightforward to introduce search into our framework (Guerrieri, Shimer and Wright, 2010; Chang, 2010).

There is a large literature on dynamic adverse selection models. In many cases, the authors implicitly assume that all trades must take place at one price, so there is necessarily a pooling equilibrium (e.g. Eisfeldt, 2004; Kurlat, 2009; Daley and Green, 2010; Chari, Shourideh and Zetlin-Jones, 2010). This implies that sellers choose not to sell some assets because the price is too low, and so in a sense these models also deliver illiquidity. In our model, in contrast, sellers try to sell all their assets, but most only sell with some probability in each period. Moreover, our model allows for the possibility that a seller can demand a high price for her asset, something that models which impose a uniform price cannot address. At the end of our paper, we compare our notion of equilibrium to an environment in which we impose that all trades must take place at a common price. A number of substantive results differ. For example, our equilibrium is unique while equilibrium is generally not unique in the pooling environment. In addition, public asset purchases, a policy intervention that is important in practice, are more effective in our environment than in the economy with pooling. For a detailed analysis of the effect of current and anticipated future public asset purchase programs in a pooling environment, see Chiu and Koeppl (2011).

A third approach to adverse selection assumes random matching between uninformed buyers and informed sellers and allows the buyers to make take-it-or-leave-it offers to sellers. Some buyers offer higher prices than others and the owners of high quality assets only sell when they are offered a high price. This generates an endogenous composition of sellers, which mitigates the adverse selection problem in that environment (Inderst, 2005; Camargo and Lester, 2011). Our approach to generating a separating equilibrium is fundamentally different in that it does not depend on an endogenous composition of sellers. We highlight this by assuming in our simplest model that the fraction of individuals who are sellers and the fraction of assets owned by those individuals is constant and exogenous.

This paper builds on our previous work with Randall Wright (Guerrieri, Shimer and Wright, 2010). It also complements a contemporaneous paper by Chang (2010). There are a number of small differences between that paper and this one. For example, we look at an environment in which individuals may later want to resell assets that they purchase today. This means that buyers care about the liquidity of the asset and so liquidity affects the equilibrium price-dividend ratio. It follows that interventions in the market which boost liquidity may also raise asset prices. We allow individuals to hold multiple assets, although that turns out to be inessential for our analysis. We also focus explicitly on a general equilibrium environment, allowing for the possibility that buyers may be driven to a corner in which they do not consume anything. This is essential for our model to generate a flight to quality. Still, both papers leverage our earlier research to study separating equilibria in a dynamic adverse selection environment.

Our notion of liquidity builds on DeMarzo and Duffie (1999), who study optimal security design by an issuer with private information. That paper shows that the issuer may commit to retain some ownership of the security in order to signal that it is of high quality. We show that in an equilibrium environment, there is no need for sellers to make such commitments. Instead, when the seller of a high quality asset demands a high price, the market ensures that the seller retains ownership with some probability by rationing sales at that price.

This paper proceeds as follows. Section 2 describes our basic model. Section 3 describes the individual's problem and shows how to express it recursively. Section 4 defines equilibrium and establishes existence and uniqueness. Section 5 provides closed-form solutions for a version of the model with a continuum of assets. Section 6 extends the model to have persistent preference shocks and then shows that the frictions survive in the continuous time limit. Section 7 discusses how our model can generate fire sales following the revelation of some information and how illiquidity and insolvency can be alleviated through an asset purchase program, although the program necessarily loses money. Section 8 compares the implications of our equilibrium concept to an environment in which a pooling equilibrium is imposed.

2 Model

There is a unit measure of risk-neutral individuals. In each period t, they can be in one of two states, $s_t \in \{l, h\}$, which determines their discount factor β_{s_t} between periods t and $t+1$. We assume $0 < \beta_l < \beta_h < 1$. The preference shock is independent across individuals and for now we assume that it is also independent over time. Thus π_s denotes the probability that an individual is in state $s \in \{l, h\}$ in any period, and it is also the fraction of individuals who are in state s in any period. For any particular individual, let $s^t \equiv \{s_0, \ldots, s_t\}$ denote the history of states through period t.

There is a finite number of different types of assets, indicated by $j \in \{1, \ldots, J\}$. Assets are perfectly durable and so their supply is fixed; let K_j denote the measure of type j assets in the economy. Each type j asset produces δ_j units of a homogeneous, nondurable consumption good each period, and so aggregate consumption $\sum_{j=1}^{J} \delta_j K_j$ is fixed. Without loss of generality, assume that higher type assets produce more of the consumption good, $0 \leq \delta_1 < \cdots < \delta_J$. The assumption that there is a finite number of asset types simplifies our notation, but in Section 5, we discuss the limiting case with a continuum of assets.

We are interested in how a market economy allocates consumption across individuals. For the remainder of the paper, we refer to the assets as "trees" and the consumption good as "fruit." The timing of events within period t is as follows:

1. each individual i owns a vector $\{k_{i,j}\}_{j=1}^{J}$ of trees which produce fruit;

2. each individual's discount factor between periods t and $t+1$ is realized;

3. individuals trade trees for fruit in a competitive market;

4. individuals consume the fruit that they hold.

We require that each individual's consumption is nonnegative in every period and we do not allow any other trades, e.g. contingent claims against shocks to the discount factor. In addition, we assume that only the owner of a tree can observe its quality, creating an adverse selection problem. Key to our equilibrium concept, which we discuss below, is that the buyer of a tree may be able to infer its quality from the price at which it is sold. Finally, we assume that an individual's discount factor is observable and impose that only individuals with low discount factors may sell trees and henceforth call them "sellers." For symmetry, we refer to individuals with high discount factors as buyers. This configuration is reasonable in the sense that, absent an adverse selection problem, individuals with high discount factors would buy trees from individuals with low discount factors, transferring consumption from those with a high intertemporal marginal rate of substitution to those with a low one.

We now describe the competitive fruit market more precisely. After trees have borne fruit, a continuum of markets distinguished by their positive price $p \in \mathbb{R}_+$ may open up. Each buyer may take his fruit to any market (or combination of markets), attempting to purchase trees in that market. Each seller may take his trees to any market (or combination of markets) attempting to sell trees in that market.

All individuals have rational beliefs about the ratio of buyers to sellers in all markets. Let $\Theta(p)$ denote the ratio of the amount of fruit brought by buyers to a market p, relative to the cost of purchasing all the trees in that market at a price p. If $\Theta(p) < 1$, there is not enough fruit to purchase all the trees offered for sale in the market, while if $\Theta(p) > 1$, there is more than enough. A seller believes that if he brings a tree to a market p, it will sell with probability $\min\{\Theta(p), 1\}$. That is, if there are excess trees in the market, the seller believes that he will succeed in selling it only probabilistically. Likewise, a buyer who brings p units of fruit to market p believes that he will buy a tree with probability $\min\{\Theta(p)^{-1}, 1\}$. If there is excess fruit in the market, he will be rationed. A seller who is rationed keeps his tree until the following period, while a buyer who is rationed must eat his fruit.

Individuals also have rational beliefs about the types of tree sold in each market. Let $\Gamma(p) \equiv \{\gamma_j(p)\}_{j=1}^J \in \Delta^J$ denote the probability distribution over trees available for sale in a market p, where Δ^J is the J-dimensional unit simplex.[2] Buyers expect that, conditional on buying a tree at a price p, it will be a type j tree with probability $\gamma_j(p)$. Buyers only learn the quality of the tree that they have purchased after giving up their fruit. They have no recourse if unsatisfied with the quality.

Although trade does not happen at every price p, the functions Θ and Γ are not arbitrary. Instead, if $\Theta(p) < \infty$ (the buyer-seller ratio is finite) and $\gamma_j(p) > 0$ (a positive fraction of the trees for sale are of type j), sellers must find it weakly optimal to sell type j trees at

[2]That is, $\gamma_j(p) \geq 0$ for all j and $\sum_{j=1}^J \gamma_j(p) = 1$.

price p. Without this restriction on beliefs, there would be equilibria in which, for example, no one pays a high price for a tree because everyone believes that they will only purchase low quality trees at that price. We define equilibrium precisely in Section 4 below.

We assume throughout this paper that the endogenous functions Θ and Γ are constant over time, so the environment is in a sense stationary. This restriction seems natural to us, and indeed we are able to prove existence and uniqueness of an equilibrium with this property. Key to this result is that, although the distribution of tree holdings across individuals evolves over time, the fraction of type j trees held by individuals with a high discount factor is necessarily a constant π_h at the start of every period because preferences are independently and identically distributed over time. Despite being skeptical that such equilibria exist, we are unable to prove that there is no equilibrium in which Θ and Γ change deterministically or stochastically over time.

3 Individual's Problem

Each individual starts off at time 0 with some vector of tree holdings $\{k_j\}_{j=1}^J$ and preference state $s \in \{l, h\}$. In each subsequent period t and history of preference shocks s^t, he decides how many trees to attempt to buy or sell at every possible possible price p, recognizing that he may be rationed at some prices and that the price may affect the quality of the trees that he buys. Let $V_s^*(\{k_j\})$ denote the supremum of the individual's expected lifetime utility over feasible policies, given initial preferences s and tree holdings $\{k_j\}$. In Appendix A, we characterize this value explicitly and prove that it is linear in tree holdings: $V_s^*(\{k_j\}) \equiv \sum_{j=1}^J v_{s,j} k_j$ for some positive numbers $v_{s,j}$. This is a consequence of the linearity of both the individual's objective function and the constraints that he faces.

In addition, we prove that the marginal value of tree holdings satisfy relatively simple recursive problems. A seller solves

$$v_{l,j} = \delta_j + \max_p \big(\min\{\Theta(p), 1\} p + (1 - \min\{\Theta(p), 1\}) \beta_l \bar{v}_j \big), \tag{1}$$

where

$$\bar{v}_j \equiv \pi_h v_{h,j} + \pi_l v_{l,j}. \tag{2}$$

The individual earns a dividend δ_j from the tree and also gets p units of fruit if he manages to sell the tree at the chosen price p. Otherwise he keeps the tree until the following period. Note that there is no loss of generality in assuming that a seller always tries to sell all his trees, since he can always offer them at a high price such that this is optimal, $p > \beta_l \bar{v}_j$. Of course, at such a high price, he may be unable to sell it, $\Theta(p) = 0$, in which case the outcome is the same as holding onto the tree.

Similarly, a buyer solves

$$v_{h,j} = \max_p \left(\min\{\Theta(p)^{-1}, 1\} \frac{\delta_j}{p} \beta_h \sum_{j'} \gamma_{j'}(p) \bar{v}_{j'} + (1 - \min\{\Theta(p)^{-1}, 1\}) \delta_j \right) + \beta_h \bar{v}_j.$$

A type j tree delivers δ_j of fruit, which the buyer uses in an attempt to purchase trees at an optimally chosen price p. If he succeeds, he buys δ_j/p trees of unknown quality, type j' with probability $\gamma_{j'}(p)$, while if he fails he consumes the fruit. Finally, he gets the continuation value of the tree in the next period. Again, note that a buyer always finds it weakly optimal to attempt to purchase a tree at a sufficiently low price p, rather than simply consuming the fruit without attempting to purchase a tree. We therefore do not explicitly incorporate this last option in the value function. Since the maximand is multiplicative in δ_j, we can equivalently write the buyer's value function as

$$v_{h,j} = \delta_j \lambda + \beta_h \bar{v}_j, \tag{3}$$

where

$$\lambda \equiv \max_p \left(\min\{\Theta(p)^{-1}, 1\} \frac{\beta_h \sum_{j=1}^J \gamma_j(p) \bar{v}_j}{p} + \left(1 - \min\{\Theta(p)^{-1}, 1\}\right) \right). \tag{4}$$

The variable λ is the value of a unit of fruit to a buyer. If $\lambda = 1$, a unit of fruit is simply worth its consumption value, and so buyers find it weakly optimal to consume their fruit. But we may have $\lambda > 1$ in equilibrium, so buyers strictly prefer to use their fruit to purchase trees.

Proposition 1 *Let $\{v_{s,j}\}$ and λ be positive-valued numbers that solve the Bellman equations* (1)*,* (3)*, and* (4) *for $s = l, h$. Then $V_s^*(\{k_j\}) \equiv \sum_{j=1}^J v_{s,j} k_j$ for all $\{k_j\}$.*

The proof is in appendix A. Note that for some choices of the functions Θ and Γ, there is no positive-valued solution to the Bellman equations. In this case, the price of trees is so low that it is possible for an individual to obtain unbounded utility and there is no solution to the individual's problem. Not surprisingly, this cannot be the case in equilibrium.

4 Equilibrium

4.1 Partial Equilibrium

We are now ready to define equilibrium. We do so in two steps. First, we define an equilibrium where the buyer's value of fruit λ is fixed, which we call "partial equilibrium". Then, we turn to the complete definition of a competitive equilibrium, where the value of λ is endogenous and ensures that the fruit market clears.

Definition 1 *A partial equilibrium with adverse selection for fixed $\lambda \geq 1$ is a pair of vectors $\{v_{h,j}\} \in \mathbb{R}_+^J$ and $\{v_{l,j}\} \in \mathbb{R}_+^J$, functions $\Theta : \mathbb{R}_+ \mapsto [0, \infty]$ and $\Gamma : \mathbb{R}_+ \mapsto \Delta^J$, a set of prices $\mathbb{P} \subset \mathbb{R}_+$, and a nondecreasing function $F : \mathbb{R}_+ \mapsto [0, 1]$ satisfying the following conditions:*

1. *Sellers' Optimality: for all $j \in \{1, \ldots, J\}$, $v_{l,j}$ solves (1) where $\bar{v}_j$ is defined in (2);*
2. *Rational Beliefs: for all $j \in \{1, \ldots, J\}$ and for all p with $\Theta(p) < \infty$ and $\gamma_j(p) > 0$,*
3. *Buyers' Optimality: for all $j \in \{1, \ldots, J\}$, $v_{h,j}$ solves (3) where λ is defined in (4) and $\bar{v}_j$ in (2);*
4. *Active Markets: $p \in \mathbb{P}$ only if it solves the maximization problem on the right-hand side of expression (4);*

$$v_{l,j} = \delta_j + \min\{\Theta(p), 1\}p + (1 - \min\{\Theta(p), 1\})\beta_l \bar{v}_j;$$

5. *Consistency of Supply with Beliefs: for all $j \in \{1, \ldots, J\}$,*

$$\frac{K_j}{\sum_{j'} K_{j'}} = \int_{\mathbb{P}} \gamma_j(p) dF(p).$$

We use $F(p)$ to denote the fraction of trees that are offered for sale at a price less than or equal to p and $\mathbb{P}$ to denote the support of this distribution function. *Sellers' Optimality* requires that sellers choose an optimal price for selling each type of tree, given the ease of trade. *Rational Beliefs* imposes that if individuals expect some type j trees to be for sale at price p, sellers must be willing to sell trees at this price, so the value of selling a type j tree at price p must be exactly $v_{l,j}$. *Buyers' Optimality* states that buyers choose an optimal price to buy trees, given the ease of trade and the composition of trees for sale at each price. *Active Markets* imposes that if there is trade at a price p, this must be an optimal price for buying trees. Finally, *Consistency of Supply with Beliefs* imposes that the share of sellers' trees that are of type j is equal to the fraction of type j trees among those offered for sale.

Definition 2 *For given λ, a solution to problem (P_j) is a vector $(v_{l,j}, \bar{v}_j, \theta_j, p_j)$ that solves* *the following Bellman equation*

$$v_{l,j} = \delta_j + \max_{p,\theta}\big(\min\{\theta,1\}p + (1-\min\{\theta,1\})\beta_l \bar{v}_j\big)$$

$$s.t.\ \lambda \le \min\{\theta^{-1},1\}\frac{\beta_h \bar{v}_j}{p} + \big(1-\min\{\theta^{-1},1\}\big), \tag{5}$$

$$and\ v_{l,j'} \ge \delta_{j'} + \min\{\theta,1\}p + (1-\min\{\theta,1\})\beta_l \bar{v}_{j'}\ for\ all\ j' < j \tag{6}$$

with

$$\bar{v}_j = \pi_h(\delta_j\lambda + \beta_h\bar{v}_j) + \pi_l v_{l,j}.$$

We are interested in solving the sequence of problems $(P) \equiv \{(P_1), \ldots, (P_J)\}$. To do so, start with type 1 trees. Constraint (6) disappears from Problem (P_1), and so we can solve directly for $v_{l,1}$ and $\bar{v}_1$, as well as the optimal policy p_1 and θ_1. Standard arguments ensure that the maximized value is unique if $\lambda \ge 1$. In general, for Problem (P_j), the first constraint and the constraint of excluding type $j-1$ trees binds, which uniquely determines p_j and θ_j as well as $v_{l,j}$ and $v_{h,j}$. The following Lemma states this claim formally.

Lemma 1 *For fixed $\lambda \in [1, \beta_h/\beta_l]$, the solution to the sequence of problem (P) has $v_{l,j} > v_{l,j-1}$, $\bar{v}_j > \bar{v}_{j-1}$, $p_j > p_{j-1}$, and $\theta_j \le \min\{\theta_{j-1}, 1\}$ for $j \ge 2$, and is the unique such solution to the system of equations*

$$\begin{aligned} v_{l,j} &= \delta_j + \min\{\theta_j,1\}p_j + (1-\min\{\theta_j,1\})\beta_l\bar{v}_j \\ \bar{v}_j &= \pi_h(\delta_j\lambda + \beta_h\bar{v}_j) + \pi_l v_{l,j} \\ p_j &= \frac{\beta_h\bar{v}_j}{\lambda} \end{aligned}$$

for all j, with $\begin{cases} \theta_1 \ge 1 & \lambda = 1 \\ \theta_1 = 1 \quad \text{if} & \lambda \in (1, \beta_h/\beta_l) \\ \theta_1 \le 1 & \lambda = \beta_h/\beta_l \end{cases}$ *and $v_{l,j-1} = \delta_{j-1} + \theta_j p_j + (1-\theta_j)\beta_l\bar{v}_{j-1}$ for $j \ge 2$.*

We focus on values of λ between 1 and β_h/β_l because these are the relevant ones for equilibrium. One could, however, also characterize the solution to problem (P) for $\lambda > \beta_h/\beta_l$; it would have $\theta_j = 0$ for all j.

Proposition 2 *Fix $\lambda \in [1, \beta_h/\beta_l]$. There exists a partial equilibrium and any partial equilibrium is given by the solution to problem (P). More precisely:*

- *Existence: Take any $\{v_{l,j}, \bar{v}_j, \theta_j, p_j\}$ that solves problem (P). Then there exists a partial equilibrium $(v_h, v_l, \Theta, \Gamma, \mathbb{P}, F)$ where $\Theta(p_j) = \theta_j$, $\gamma_j(p_j) = 1$, $v_h = \{\delta_j\lambda + \beta_h\bar{v}_j\}$, $v_l = \{v_{l,j}\}$, $\mathbb{P} = \{p_j\}$, and $dF(p_j) = K_j/\sum_{j'} K_{j'}$.*

- *Uniqueness: Take any partial equilibrium* $(v_h, v_l, \Theta, \Gamma, \mathbb{P}, F)$*. For all* j*, there exists a* $p_j \in \mathbb{P}$ *with* $\gamma_j(p_j) > 0$*. If also* $\Theta(p_j) > 0$*, then* $(v_{l,j}, \bar{v}_j, \Theta(p_j), p_j)$ *solves problem* (P_j)*.*

The proof in the appendix gives a complete characterization of the partial equilibrium, including the entire functions Θ and Γ. Since we proved in Lemma 1 that the solution to problem (P) is unique, except possibly for the value of θ_1, this essentially proves uniqueness of the partial equilibrium.

4.2 Competitive Equilibrium

We now turn to a full competitive equilibrium in which λ is endogenous:

Definition 3 *A competitive equilibrium with adverse selection is a number* $\lambda \in [1, \beta_h/\beta_l]$*, a pair of vectors* $\{v_{h,j}\} \in \mathbb{R}_+^J$ *and* $\{v_{l,j}\} \in \mathbb{R}_+^J$*, functions* $\Theta : \mathbb{R}_+ \mapsto [0, \infty]$ *and* $\Gamma : \mathbb{R}_+ \mapsto \Delta^J$*, a set of prices* $\mathbb{P} \subset \mathbb{R}_+$*, and a nondecreasing function* $F : \mathbb{R}_+ \mapsto [0, 1]$ *satisfying the following conditions:*

1. $(\{v_{h,j}\}, \{v_{l,j}\}, \Theta, \Gamma, \mathbb{P}, F)$ *is a partial equilibrium with adverse selection for fixed* λ*; and*
2. *the fruit market clears:* $\pi_h \sum_{j=1}^{J} \delta_j K_j = \pi_l \left(\sum_{j=1}^{J} K_j \right) \int_{\mathbb{P}} \Theta(p) p dF(p)$*.*

A competitive equilibrium is a partial equilibrium plus the market clearing condition that states that the fruit brought to market by buyers is equal to the value of trees brought to the market by sellers. Recall from Proposition 2 that $dF(p_j) = K_j / \sum_{j'} K_{j'}$ in partial equilibrium, where p_j is the equilibrium price of type j trees. The market clearing condition therefore reduces to

$$\pi_h \sum_{j=1}^{J} \delta_j K_j = \pi_l \sum_{j=1}^{J} \Theta(p_j) p_j K_j.$$

The left hand side is the fruit held by buyers at the start of the period, while each term in the right hand side is the equilibrium cost of purchasing a particular type of tree multiplied by the buyer-seller ratio for that tree.

Proposition 3 *A competitive equilibrium* $(\lambda, v_h, v_l, \Theta, \Gamma, \mathbb{P}, F)$ *exists and is unique.*

The proof shows that an increase in the value of fruit to a buyer λ drives down the amount of fruit that sellers expect to get from selling any type j tree, that is, $p_j \Theta(p_j)$. Indeed, in the limit when $\lambda = \beta_h/\beta_l$, $\Theta(p_j) = 0$ for all $j > 1$, and so trade breaks down in all but the worst type of tree. At the opposite limit of $\lambda = 1$, buyers are indifferent about purchasing trees and so $\Theta(p_1) > 1$ and buyers are rationed. By varying λ, we find the unique value at which the fruit market clears.

In general, we can distinguish between three cases, each of which is generic in the parameter space. We show in the proof of Proposition 3 that if there are very few sellers, $\pi_l < \underline{\pi}$, the unique equilibrium has $\lambda = 1$ and buyers consume some of their fruit. Conversely, if there are many sellers, $\pi_l > \bar{\pi}$, the unique equilibrium has $\lambda = \beta_h/\beta_l$ and there is only a market in the worst type of tree. At intermediate values of π_l, $\beta_h/\beta_l > \lambda > 1$, there is a market for every type of tree, and buyers use all their fruit to purchase trees. The thresholds satisfy $1 > \bar{\pi} > \underline{\pi} > 0$ and depend on all the other model parameters.

5 Continuous Types of Trees

We have assumed so far that there are only a finite number of types of trees. It is conceptually straightforward to extend our analysis to an environment with a continuum of trees. This is useful because it shows that the behavior of the economy is not particularly sensitive to the number of types of trees, but rather it depends on the support of the tree distribution.

The only change in our environment is that we assume the tree distribution is dense on $(\underline{\delta}, \bar{\delta})$, where $0 \leq \underline{\delta} < \bar{\delta} \leq \infty$. Let $G(\delta)$ denote the cumulative distribution of trees on this support. We similarly let $v_l(\delta)$, $v_h(\delta)$, and $\bar{v}(\delta)$ denote the value to a seller, the value to a buyer, and the expected value of a tree that bears δ units of fruit per period. These satisfy the analogs of equations (1)–(3). Definition 1 (partial equilibrium) and Definition 3 (competitive equilibrium) change only to reflect this new notation.[3] We omit these formalities in the interest of space.

We find that in equilibrium, the price of the lowest quality tree is

$$\underline{p} = \frac{\underline{\delta}\beta_h(\pi_l + \lambda\pi_h)}{\lambda - \beta_h(\pi_l + \lambda\pi_h)}. \tag{7}$$

For $p < \underline{p}$, $\Theta(p) = \infty$ and $\Gamma(p)$ is defined arbitrarily. For $p > \underline{p}$,

$$\Theta(p) = \left(\underline{p}/p\right)^{\frac{\beta_h}{\beta_h - \beta_l\lambda}}, \tag{8}$$

[3] One minor modification is the market clearing condition when $\lambda = 1$. In the economy with finitely many types of trees, we used $\Theta(p_1) \geq 1$ to ensure that buyers brought all their trees to the market even when $\lambda = 1$. Here it is easier to allow buyers to consume a positive fraction of their fruit and impose $\Theta(P(\underline{\delta})) = 1$.

while a different type of tree $\delta = D(p)$ is offered at each price p, where

$$D(p) = p\left(\frac{\lambda + (\beta_h - \lambda\beta_l)(1-\Theta(p))\pi_l}{\beta_h(\pi_l + \lambda\pi_h)} - 1\right). \tag{9}$$

These equations hold as long as $D(p) \leq \bar{\delta}$. For higher prices, $\Theta(p)$ is pinned down by the indifference curve of the seller of a type $\bar{\delta}$ tree and $D(p) = \bar{\delta}$. This pins down a partial equilibrium for fixed $\lambda \in [1, \beta_h/\beta_l]$.

We also find that the competitive equilibrium is unique and always has $\lambda < \beta_h/\beta_l$. It satisfies

$$\pi_h \int_{\underline{\delta}}^{\bar{\delta}} \delta dG(\delta) \geq \pi_l \int_{\underline{\delta}}^{\bar{\delta}} \Theta(P(\delta))P(\delta)dG(\delta) \text{ with equality if } \lambda > 1, \tag{10}$$

where $P(\delta)$ is the equilibrium price of a type δ tree, so $D(P(\delta)) \equiv \delta$. If this holds as an inequality, the difference is the measure of fruit consumed by buyers. We can prove directly from the functional form for Θ and P that an increase in λ reduces the right hand side of this inequality, ensuring that the competitive equilibrium is unique. Indeed, as λ converges to β_h/β_l, the right hand side converges to 0, ruling out the possibility of an equilibrium in which λ takes on this limiting value. We summarize these results in a proposition:

Proposition 4 *Equations* (7)–(10) *uniquely describe a competitive equilibrium when the support of the tree distribution is dense on* $(\underline{\delta}, \bar{\delta})$. *This is the unique limit of the economy with a finite number of trees.*

We believe that this is also the unique equilibrium of the limiting economy, but our approach to establishing uniqueness—solving a sequence of problems (P)—does not easily extend to an economy with uncountably many types of trees.

6 Persistent Shocks and Continuous Time

Our model explains how adverse selection can generate trading frictions, in the sense that a tree only sells with a certain probability each period. But suppose that the time between periods is negligible. Will the trading frictions become negligible as well? We argue in this section that they will not. Instead, equilibrium requires that a real amount of calendar time elapse before a high quality tree is sold.

To show this, we consider the behavior of the economy when the number of periods per unit of calendar time increases without bound. That is, we take the limit of the economy as the discount factors converge to 1, holding fixed the ratio of discount rates $(1-\beta_h)/(1-\beta_l)$ and the present value of dividends $\delta_j/(1-\beta_s)$. But as we take this limit, we also want to avoid changing the stochastic process of shocks. With i.i.d. shocks and very short time periods, there is almost no difference in preferences between high and low types of individuals and so the gains from trade become negligible. We therefore first introduce persistent shocks into the model and then prove that as the period length shortens, the probability of sale per period falls to zero, while the probability of sale per unit of calendar time converges to a well-behaved number.

6.1 Persistent Shocks

Assume now that $s_t \in \{l, h\}$ follows a first order stochastic Markov process and let $\pi_{ss'}$ denote the probability that the state next period is s' given that the current state is s. A partial equilibrium with a fixed value of $\lambda \geq 1$ is still characterized by a pair of functions $\{v_{s,j}\} \in \mathbb{R}_+^{2J}$ that represent the value of an individual who starts a period in preference state s holding a type j tree; a function $\Theta : \mathbb{R}_+ \mapsto [0, \infty]$ representing the buyer-seller ratio at an arbitrary price p; a function $\Gamma : \mathbb{R}_+ \mapsto \Delta^J$ representing the distribution of tree types available at price p; a set of transaction prices $\mathbb{P} \subset \mathbb{R}_+$; and a nondecreasing function $F : \mathbb{R}_+ \mapsto [0, 1]$ representing the share of trees available at a price less than or equal to p. The definition of partial equilibrium is analogous to definition 1 for the i.i.d. case, except for the obvious change in the continuation value:

$$v_{l,j} = \delta_j + \max_p \big(\min\{\Theta(p), 1\}p + (1 - \min\{\Theta(p), 1\})\beta_l(\pi_{ll}v_{l,j} + \pi_{lh}v_{h,j})\big), \tag{1'}$$

$$v_{h,j} = \delta_j\lambda + \beta_h(\pi_{hl}v_{l,j} + \pi_{hh}v_{h,j})\big), \tag{3'}$$

where

$$\lambda \equiv \max_p \left(\min\{\Theta(p)^{-1}, 1\} \frac{\beta_h \sum_{j=1}^J \gamma_j(p)(\pi_{hl}v_{l,j} + \pi_{hh}v_{h,j})}{p} + \big(1 - \min\{\Theta(p)^{-1}, 1\}\big) \right). \tag{4'}$$

We omit the formal definition, which simply substitutes these expressions for their i.i.d. analogs. The characterization of partial equilibrium and proof that it exists and is unique is similarly unchanged. In equilibrium, type j trees sell for a price p_j satisfying the buyers' indifference condition

$$p_j = \frac{\beta_h(\pi_{hl}v_{l,j} + \pi_{hh}v_{h,j})}{\lambda},$$

while the condition for excluding type $j-1$ trees from the market pins down the sale probability θ_j when $j \geq 2$

$$\theta_j\big(p_j - \beta_l(\pi_{ll}v_{l,j-1} + \pi_{lh}v_{h,j-1})\big) = \min\{\theta_{j-1}, 1\}\big(p_{j-1} - \beta_l(\pi_{ll}v_{l,j-1} + \pi_{lh}v_{h,j-1})\big).$$

These equations pin down the value functions, prices, and trading frictions given λ.

In the model with idiosyncratic shocks, we found that the value of fruit to a high discount factor individual, λ, always lies in the interval $[1, \beta_h/\beta_l]$. With persistent shocks, the lower bound, which ensures that high discount factor individuals are willing to buy trees, $p_j \leq \beta_h(\pi_{hl}v_{l,j} + \pi_{hh}v_{h,j})$, is unchanged. However, the upper bound, which ensures that low discount factor individuals are willing to sell trees, $p_j \geq \beta_l(\pi_{ll}v_{l,j} + \pi_{lh}v_{h,j})$, is given by the larger root of

$$\beta_h(\lambda - (\lambda - 1)\pi_{hl}) = \beta_l\lambda(\lambda - (\lambda - 1)\pi_{ll}).$$

We denote this upper bound by $\bar{\lambda}$. It always exceeds 1 and $\bar{\lambda} > \beta_h/\beta_l$ if and only if shocks are persistent, $\pi_{ll} > \pi_{hl}$.

The definition of a competitive equilibrium with persistent shocks is also complicated by endogeneity of the distribution of tree holdings. In the i.i.d. case, high discount factor individuals start each period holding a fraction $\pi_h K_j$ type j trees, but this is not true with persistent shocks. Instead, let μ_j denote the measure of type j trees held by high discount factor individuals at the start of a period. In steady state, this satisfies

$$\mu_j = \pi_{hh}\left(\mu_j + \sigma_j\right) + \pi_{lh}\left(K_j - \mu_j - \sigma_j\right),$$

where σ_j is the measure of type j trees purchased by high discount factor individuals each period. High discount factor individuals hold $\mu_j + \sigma_j$ type j trees at the end of each period, while the rest are held by low discount factor individuals. Multiplying by the appropriate preference transition probabilities delivers the measure held by high discount factor individuals at the start of the following period. To solve for μ_j, we first need to compute the measure of trees sold each period, σ_j. This is the product of the measure of trees for sale times the average sale probability weighted by the fraction of trees that are of type j at an arbitrary price p:

$$\sigma_j = \left(\sum_{j'}(K_{j'} - \mu_{j'})\right)\int_{\mathbb{P}} \min\{\Theta(p), 1\}\gamma_j(p)dF(p).$$

Alternatively, Consistency of Supplies with Beliefs implies

$$\frac{K_j - \mu_j}{\sum_{j'}(K_{j'} - \mu_{j'})} = \int_{\mathbb{P}} \gamma_j(p)dF(p),$$

and so we can rewrite the measure sold as

$$\sigma_j = (K_j - \mu_j)\frac{\int_{\mathbb{P}} \min\{\Theta(p), 1\}\gamma_j(p)dF(p)}{\int_{\mathbb{P}} \gamma_j(p)dF(p)},$$

the product of the measure of trees for sale and the average sale probability. Use this to solve for μ_j:

$$\mu_j = \frac{\pi_{lh} + (\pi_{hh} - \pi_{lh})\frac{\int_{\mathbb{P}} \min\{\Theta(p),1\}\gamma_j(p)dF(p)}{\int_{\mathbb{P}} \gamma_j(p)dF(p)}}{1 - (\pi_{hh} - \pi_{lh})\left(1 - \frac{\int_{\mathbb{P}} \min\{\Theta(p),1\}\gamma_j(p)dF(p)}{\int_{\mathbb{P}} \gamma_j(p)dF(p)}\right)}K_j. \tag{11}$$

If $\pi_{hh} = \pi_{lh}$, this reduces to $\mu_j = \pi_{lh}K_j = \pi_{hh}K_j$, but if shocks are persistent, $\pi_{hh} > \pi_{lh}$, then μ_j is increasing in the measure of type j trees that are sold each period.

We are now in a position to define equilibrium:

Definition 4 *A stationary competitive equilibrium with adverse selection and persistent shocks is a number $\lambda \in [1, \bar{\lambda}]$, a pair of vectors $\{v_{h,j}\} \in \mathbb{R}_+^J$ and $\{v_{l,j}\} \in \mathbb{R}_+^J$, functions $\Theta : \mathbb{R}_+ \mapsto [0, \infty]$ and $\Gamma : \mathbb{R}_+ \mapsto \Delta^J$, a set of prices $\mathbb{P} \subset \mathbb{R}_+$, a nondecreasing function $F : \mathbb{R}_+ \mapsto [0, 1]$, and measures $\mu_j \in [0, K_j]$ satisfying the following conditions:*

1. *$(\{v_{h,j}\}, \{v_{l,j}\}, \Theta, \Gamma, \mathbb{P}, F)$ is a partial equilibrium with adverse selection and persistent shocks for fixed λ;*
2. *the fruit market clears: $\displaystyle\sum_{j=1}^{J} \delta_j\mu_j = \left(\sum_{j=1}^{J}(K_j - \mu_j)\right)\int_{\mathbb{P}} \Theta(p)pdF(p)$; and*
3. *measures are consistent with trades: μ_j satisfies equation* (11).

If there are a continuum of types trees, we can again obtain closed-form solutions. In particular, arguments analogous to those in Proposition 4 imply

$$\Theta(p) = \frac{\lambda(1 - \beta_l(\pi_{ll} - \pi_{hl})) - (\lambda - 1)\pi_{hl}}{(\lambda - (\lambda - 1)\pi_{hl})(p/\underline{p})^{\frac{\beta_h(\lambda(1-\beta_l(\pi_{ll}-\pi_{hl}))-(\lambda-1)\pi_{hl})}{\beta_h(\lambda-(\lambda-1)\pi_{hl})-\beta_l\lambda(\lambda-(\lambda-1)\pi_{ll})}} - \beta_l\lambda(\pi_{ll} - \pi_{hl})}. \tag{8'}$$

Similarly, the type of asset sold at price p satisfies

$$D(p) = p\left(\frac{\lambda + (\beta_h\pi_{hl} - \lambda\beta_l\pi_{ll})(1 - \Theta(p))}{\beta_h(\lambda - (\lambda - 1)\pi_{hl} + (1 - \Theta(p))\beta_l\lambda(\pi_{hl} - \pi_{ll}))} - 1\right). \tag{9'}$$

These expressions generalize equations (8) and (9) to the model with persistent shocks. Finally, the share of trees that are of type δ or less and are held by high discount factor individuals is

$$G_h(\delta) = \int_{\underline{\delta}}^{\delta} \frac{\pi_{lh} + (\pi_{hh} - \pi_{lh})\Theta(P(\delta'))}{1 - (\pi_{hh} - \pi_{lh})(1 - \Theta(P(\delta')))}dG(\delta'), \tag{11'}$$

where again $P(\delta)$ is the inverse of $D(p)$.

We do not prove existence and uniqueness of equilibrium in this environment. For starters, extending the proof of Proposition 3 is cumbersome because the measures μ_j are endogenous and depend on λ. But this can easily be handled using the closed-form solutions when there are a continuum of types of trees. More importantly, such a proof would only establish existence and uniqueness of a stationary competitive equilibrium, not that there is a unique equilibrium for arbitrary initial conditions. The distinction is important because μ_j is a payoff-relevant state variable in the model with persistent shocks. Given an initial value of the vector $\{\mu_j\}$, subsequent trades determine the evolution of this vector, which in turn determines the evolution of the value of fruit to a buyer λ. We have not characterized a partial equilibrium with time-varying λ, indeed we have not even introduced notation that would allow us to do so. Therefore we cannot discuss the full set of potentially nonstationary equilibria in this environment. Nevertheless, we believe that our analysis of stationary equilibria is an important first step.

6.2 Continuous Time Limit

We are now in a position to consider the continuous time limit of this model. For fixed $\Delta > 0$, define discount rates ρ_s and transition rates q_{hl} and q_{lh} as

$$\rho_s = \frac{1-\beta_s}{\Delta},\ q_{hl} = \frac{\pi_{hl}}{\Delta},\ \text{and } q_{lh} = \frac{\pi_{lh}}{\Delta}.$$

Also assume a type δ tree produces $\delta\Delta$ fruit per period. We interpret $1/\Delta$ as the number of periods within a unit of calendar time. With fixed values of ρ_s, q_{hl}, and q_{lh}, the limit as $\Delta \to 0$ (and so $\beta_s \to 1$ and π_{hl} and $\pi_{lh} \to 0$) then corresponds to the continuous time limit of the model. We find that in this limit, $\Theta(p) \to 0$ but the sale rate per unit of time converges to a number:

$$\alpha(p) \equiv \lim_{\Delta\to 0}\frac{\Theta(p)}{\Delta} = \frac{\rho_l + q_{lh} + q_{hl}/\lambda}{\left(p/\underline{p}\right)^{\frac{\rho_l+q_{lh}+q_{hl}/\lambda}{\rho_l-\rho_h-(\lambda-1)(q_{lh}+q_{hl}/\lambda)}} - 1}$$

for all $p \geq \underline{p}$, while the type of tree sold at price p converges to

$$D(p) = p\left(\rho_h + \frac{q_{hl}((\lambda-1)\alpha(p) + \lambda\rho_l - \rho_h)}{q_{hl} + \lambda(q_{lh} + \rho_l + \alpha(p))}\right).$$

In particular, the worst type of tree has dividend per unit of calendar time $\underline{\delta} = D(\underline{p})$ and no resale risk, $\alpha(\underline{p}) = \infty$. This pins down the lowest price,

$$\underline{p} = \frac{\underline{\delta}\lambda}{(\lambda - 1)q_{hl} + \lambda\rho_h}.$$

From the perspective of a seller, $\alpha(p)$ is the arrival rate of a Poisson process that permits her to sell at a price p. Equivalently, the probability that she fails to sell at a price $p > \underline{p}$ during a unit of elapsed time is $\exp(-\alpha(p))$, an increasing function of p that converges to 1 as p converges to infinity and is well-behaved in the limiting economy. One can also find the arrival rate of trading opportunities to a buyer; this is infinite if $p > \underline{p}$ and zero if $p < \underline{p}$.

To close the model, we can compute the measure of type δ trees held by high discount factor individuals, the limit of equation (11′). This gives

$$G_h(\delta) = \int_{\underline{\delta}}^{\delta} \frac{q_{lh} + \alpha(P(\delta'))}{q_{hl} + q_{lh} + \alpha(P(\delta'))} dG(\delta').$$

Substituting this into the fruit market clearing condition gives

$$\int_{\underline{\delta}}^{\bar{\delta}} \frac{\delta(q_{lh} + \alpha(P(\delta)))}{q_{hl} + q_{lh} + \alpha(P(\delta))} dG(\delta) \geq \int_{\underline{\delta}}^{\bar{\delta}} \frac{\alpha(P(\delta))P(\delta)q_{hl}}{q_{hl} + q_{lh} + \alpha(P(\delta))} dG(\delta),$$

with equality if $\lambda > 1$. The left hand side is the integral of the dividend per unit of time δ times the density $dG_h(\delta)$, i.e. the amount of fruit held by high discount factor individuals at the start of a period. The integrand on the right hand side is the product of the probability per unit of time of selling a type δ tree, $\alpha(P(\delta))$, times the price of the tree, $P(\delta)$, times the density of such trees held by low discount factor individuals, $dG(\delta) - dG_h(\delta)$. Integrating over the support of the dividend distribution gives the amount of fruit required to purchase the trees that are sold at each instant.

In equilibrium, there is a continuum of marketplaces, each distinguished by its price p. Sellers try to sell their trees in the appropriate market, while buyers bring their fruit to markets and possibly consume some of it. In all but the worst market, with price $\underline{p}$, there is always too little fruit to purchase all of the trees. That is, a stock of trees always remains in the market to be purchased by the gradual inflow of new fruit from buyers. Buyers are able to purchase trees immediately, but sellers are rationed and get rid of their trees only at a Poisson rate. Of course, a seller could immediately sell her trees for the low price $\underline{p}$, but she chooses not to do so.

More generally, the trading frictions generated by adverse selection do not disappear when the period length is short. Intuitively, it must take a real amount of calendar time to sell a tree at a high price or the owners of low quality trees would misrepresent them as being of high quality. This is in contrast to models where trading is slow because of search frictions.[4] In such a framework, the extent of search frictions governs the speed of trading and as the number of trading opportunities per unit of calendar time increases, the relevant frictions naturally disappear.

7 Discussion

Our model shows how prices and illiquidity can be used to separate trees with different qualities. High quality trees trade at a higher price but the market is less liquid. A seller could always choose to sell them at a lower price, but in equilibrium she prefers not to do so. This section explores how our model can be used to understand a financial crisis characterized by a collapse in the liquidity and price of some assets and a flight to other high quality, liquid assets. We also ask how outside intervention may increase liquidity and prices in the first type of market and restore normal prices in the second.

We focus throughout this section on the model with a continuum of assets, $\delta \in [\underline{\delta}, \bar{\delta}]$, i.i.d. preference shocks, and discrete time. The first restriction is unimportant, but the assumption that preference shocks are i.i.d. obviates the need to discuss transitional dynamics.

7.1 Fire Sales and Flight to Quality

We imagine an initial situation in which everyone believes that all trees produce δ_0 fruit per unit of time. At time 0, everyone learns that there is dispersion in the quality of trees. For example, this may correspond to the development of a technology that tells sellers which of their trees produce more fruit. In this case, average fruit production is still δ_0 but there is asymmetric information. Alternatively, the outbreak of a disease may reduce the productivity of some trees while leaving others unaffected, reducing average fruit production and creating asymmetric information. After this shock, the support of the quality distribution is now some interval $[\underline{\delta}, \bar{\delta}]$, where $\underline{\delta} < \delta_0$ and we allow $\bar{\delta} \gtreqless \delta_0$.

We first consider a partial equilibrium exercise where the value of λ is held fixed. Naturally the price of trees with $\delta < \delta_0$ falls, since these trees are known to be of lower quality than before. Moreover, the market for all trees with $\delta > \underline{\delta}$ becomes less liquid pushing down their resale value. We claim that the average price of a tree falls as long as the average tree

[4]See, for example, Duffie, Gârleanu and Pedersen (2005), Weill (2008), and Lagos and Rocheteau (2009) for models where assets are illiquid because of search frictions.

quality does not increase (or does not increase by too much). Inverting equation (9) gives

$$\begin{aligned}\int_{\underline{\delta}}^{\bar{\delta}} P(\delta) dG(\delta) &= \int_{\underline{\delta}}^{\bar{\delta}} \delta \left(\frac{\lambda + (\beta_h - \lambda\beta_l)(1 - \Theta(P(\delta)))\pi_l}{\beta_h(\pi_l + \lambda\pi_h)} - 1 \right)^{-1} dG(\delta) \\ &\leq \frac{\beta_h(\pi_l + \lambda\pi_h)}{\lambda - \beta_h(\pi_l + \lambda\pi_h)} \int_{\underline{\delta}}^{\bar{\delta}} \delta dG(\delta)\end{aligned}$$

The inequality uses $\Theta(p) < 1$ for all $p > \underline{p} = P(\underline{\delta})$. The term multiplying the integral is the price-dividend ratio in the economy where all assets are worth δ_0, and so adverse selection reduces the price-dividend ratio and hence the average price as long as the dividend does not increase by too much.

The emergence of adverse selection also has a general equilibrium effect through the value of fruit to buyers, λ. If λ did not change, lower prices and lower liquidity would reduce the amount of fruit needed to purchase trees, given by the right hand side of inequality (10). This is inconsistent with equilibrium if initially $\lambda > 1$, and so λ must fall. Equation (7) implies that a reduction in λ raises the lowest price $\underline{p}$. From equation (9) it also raises the price of better quality trees conditional on the resale probability $\Theta(P(\delta))$, while equation (8) indicates that lower λ increases the resale probability conditional on the price. Together these general equilibrium effects indicate that lower λ raises the amount of fruit needed to purchase each type of tree $\Theta(P(\delta))P(\delta)$, restoring equilibrium in the fruit market. These general equilibrium effects generate a flight to quality. To explain this requires a slight extension of our model, introducing a high quality asset. We suppose that in addition to the trees that we have already modeled—apple trees, to be concrete—there is another type of tree that is not subject to adverse selection, banana trees. Banana trees produce a known amount of fruit, apples and bananas are perfect substitutes in consumption, and either fruit can be used to purchase either type of tree. In particular, buyers value apples and bananas at a common level λ.

Absent adverse selection in the apple tree market, buyers use all their fruit to purchase all the trees, pinning down $\lambda > 1$. The emergence of adverse selection in the apple tree market reduces the amount of fruit they need to purchase the apple trees at a given value of λ. Rather than consume that fruit, we have argued that the equilibrium value of fruit λ falls to restore equilibrium in the fruit market. This raises the price and liquidity of apple trees, relative to a benchmark partial equilibrium economy with adverse selection, but the amount of fruit needed to purchase apple trees still necessarily falls. Instead, some of the excess fruit goes to purchase banana trees, driving up their price according to equation (7). This is a flight to quality.

To further understand this phenomenon, it may help to think that initially some people own only apple trees and others own only banana trees. If both classes of individuals use their fruit to purchase only that type of tree, the price-dividend ratios are identical and so everyone is fully optimizing. Now the emergence of an adverse selection problem in the apple tree market means that the "natural buyers" of apple trees—those who already own apple trees—hold more apples than they need to purchase the trees that can be sold each period. They therefore use some of their apples to purchase banana trees, driving up the price of those trees. The owners of apple trees may note that there are still enough apples to purchase all the trees available for sale at the old price-dividend ratio, but apple buyers still move towards the safe, liquid banana tree market, driving up the price of those the high-quality assets.

7.2 Asset Purchase Program

We believe our model may be useful for understanding the potential impact of an asset purchase program, such as the original vision of the Troubled Asset Relief Program in 2008 or the Public-Private Investment Program for Legacy Assets in 2009. Both of these programs were designed to remove "toxic assets" from the balance sheets of troubled financial institutions, thereby improving the solvency of the financial institutions. According to the U.S. government, this would occur not only because of the direct subsidy from the asset purchase but also through the equilibrium effects on the price and liquidity of assets that were not sold to the government. We show that this is consistent with the predictions of our model.

To model an asset purchase program, we introduce a new class of "government agents" who are initially endowed with $K_{j,0}^g$ type j trees at the start of period 0. Government agents are not utility maximizing, but instead follow instructions designed to stop low quality assets from circulating among the private agents. For this reason, we do not specify their preferences. Now, let $K_{j,0}^p$ denote the holdings of asset j of private individuals (hereafter individuals) at the start of period 0. Preferences of individuals are unchanged and we focus on the case where preference shocks are i.i.d., so the initial endowment is $\pi_h K_{j,0}^p$ for high discount factor individuals and $\pi_l K_{j,0}^p$ for low low discount factor ones.

We assume the government buys and sells trees in an effort to meet several objectives. First, it corners the market in all low quality trees sold by individuals by putting enough fruit into a market at a "government price" $\bar{p}$ so that $\Theta(\bar{p}) \geq 1$. Second, it sells high quality trees to individuals so as to keep the value of fruit to private buyers λ constant. While we could relax this restriction, doing so would require us to define a partial equilibrium with time-varying λ_t. Finally, the government does not wish to purchase all the trees from the private market, nor does it want to give its entire endowment of high quality to the private market. Instead, it wants the value of private sector tree holdings to converge to some asymptotic value.

The definition of equilibrium is unchanged, except for the behavior of government agents. Individuals with a low discount factor still set optimal prices for their trees given the sale probability $\Theta(p)$, while individuals with a high discount factor use their fruit to purchase trees and value it at $\lambda \in (1, \beta_h/\beta_l)$; we focus for simplicity on the case where λ takes on an interior value. Beliefs are rational and the set of active markets now includes the price $\bar{p}$ in addition to the prices that maximize buyers' utility. Finally, the supply of assets must be consistent with beliefs about their sale price and the fruit market must clear.

To proceed, define a threshold for assets $\bar{\delta} \equiv (\lambda - \beta_h(\lambda\pi_h + \pi_l))\bar{p}/\beta_h(\lambda\pi_h + \pi_l)$. A buyer is willing to pay $\bar{p}$ for an asset with dividend $\bar{\delta}$ if the resale market is perfectly liquid. Then if $\delta_j < \bar{\delta}$, asset j is bought only by government agents at price $\bar{p}$ in a perfectly liquid market, $\Theta(\bar{p}) \geq 1$, while if $\delta_j > \bar{\delta}$, private sector buyers purchase the asset as well at a price $p_j > \bar{p}$ and the market is illiquid, $\Theta(p_j) < 1$. It is worth stressing that some assets that sold for less than $\bar{p}$ before the program was announced may be worth more than $\bar{p}$ if completely liquid and so are not sold to the government.

Proposition 5 *Let $\bar{j} \geq 1$ be such that $\delta_{\bar{j}} < \bar{\delta} \leq \delta_{\bar{j}+1}$. If $K^g_{j,0} \geq \frac{\pi_h \sum_{j=1}^{\bar{j}} \delta_j K^p_{j,0}}{(1-\pi_h)\sum_{j=\bar{j}+1}^{J} p_j K^p_{j,0}} K^p_{j,0}$ for all $j \geq \bar{j}+1$, then government agents can implement the following policy:*

1. *buy trees of quality δ_j with $j \leq \bar{j}$ at price $\bar{p}$ so that $\Theta(\bar{p}) \geq 1$;*
2. *sell trees of quality δ_j with $j > \bar{j}$ to private agents so that $K^p_{j,t} = g_t K^p_{j,0}$, where*

$$g_t \equiv 1 + \frac{\pi_h(1-\pi_h^t)\sum_{j=1}^{\bar{j}} \delta_j K^p_{j,0}}{(1-\pi_h)\sum_{j=\bar{j}+1}^{J} p_j K^p_{j,0}}.$$

This policy keeps λ constant in equilibrium and implies $lim_{t\to\infty} K^p_{j,t} = 0$ for $j \leq \bar{j}$ and $lim_{t\to\infty} K^p_{j,t} = g_\infty K^p_{j,0}$ for $j > \bar{j}$, where $g_\infty = 1 + \frac{\pi_h \sum_{j=1}^{\bar{j}} \delta_j K^p_{j,0}}{(1-\pi_h)\sum_{j=\bar{j}+1}^{J} p_j K^p_{j,0}}$.

This program has the desired impact on prices and liquidity. To start, ignore the general equilibrium impact by keeping λ constant. By raising the price and liquidity of low quality trees, the intervention relaxes the incentive constraint. More precisely, the price and liquidity of a type $\bar{j}+1$ must satisfy the incentive constraint $\bar{p} = \theta_{\bar{j}+1}p_{\bar{j}+1} + (1-\theta_{\bar{j}+1})\beta_l \bar{v}_{\bar{j}}$, so a seller is indifferent between getting $\bar{p}$ from the government for sure or getting the lowest private sector price $p_{\bar{j}+1}$ with probability $\theta_{\bar{j}+1}$. A higher government price raises the private sector price by more than the continuation value $\beta_l \bar{v}_j$. To be consistent with the private buyers' valuation, both price and liquidity must increase. The same logic implies that this effect carries through to all higher quality trees.

The program also has a general equilibrium effect. Prior to the intervention, buyers use all their fruit to purchase all the trees sold by sellers. After the intervention, they only use some of their fruit to buy some of the trees. The net impact on λ is ambiguous. On the one hand, since more productive trees are less liquid, buyers always have more fruit from their most productive tree than is needed to purchase those trees, $\pi_h \delta_J K^p_{J,0} > \pi_l \theta_J p_J K^p_{J,0}$, and conversely for the least productive trees, $\pi_h \delta_1 K^p_{1,0} < \pi_l \theta_1 p_1 K^p_{1,0}$. This means that excluding the least productive trees and their fruit from the market will lead to a surplus of fruit. On the other hand, excluding those trees raises prices and liquidity in the market for more productive trees, leading to a shortage of fruit. If there are few low quality trees, the latter effect will dominate, and so the general equilibrium impact of the asset purchase program moderate the increase in tree purchases.

8 Pooling Environment

Much of the literature on adverse selection in financial markets assumes that all trades occur at a common price p, so the equilibrium is *pooling* (see, for example, Eisfeldt, 2004; Kurlat, 2009; Daley and Green, 2010; Chari, Shourideh and Zetlin-Jones, 2010). In contrast, we find that different types of trees never trade at the same price, so the equilibrium is separating. The source of the difference in results lies in the definitions of equilibrium. In models with pooling, the environment is set up in such a way that a seller cannot even consider selling his trees at a price different than p. In contrast, this thought experiment is central to our definition of equilibrium. In this section, we consider an alternative definition of equilibrium where we restrict all trades to occur at a common price p. The rest of the environment is exactly as in our benchmark model, but the equilibrium is necessarily pooling.[5] We show that this significantly affects several important outcomes, including the cross-sectional behavior of prices, dividends, and liquidity; the nature of fire sales; and the efficacy of asset purchase programs. Thus our notion of equilibrium is central to our results.

We look for an equilibrium in which all trades occur at a price p. Sellers (individuals with low discount factors) are able to choose whether to sell their trees at that price and buyers (individuals with high discount factors) are able to choose whether to buy trees at that price. This implies that the price must leave the marginal seller indifferent about selling his tree and it must leave the buyer indifferent about buying the average tree offered for sale.

[5] We stress that the model we have analyzed thus far in the paper has a unique equilibrium. There is no pooling equilibrium in our model.

Our definition of a pooling equilibrium embodies these requirements.

We look at a simple version of our model with i.i.d. preference shocks and a continuum of types of trees.[6] As before, we let $v_s(\delta)$ denote the marginal value of a type δ tree to an individual with discount factor β_s and λ denote the value of a unit of fruit to a buyer in excess of its consumption value. We assume that all trades occur at a common price p. Let $\zeta(\delta)$ denote the fraction of type δ trees that sellers attempt to sell at price p. This is equal to 0 if $p < \beta_l \bar{v}(\delta)$ and 1 if $p > \beta_l \bar{v}(\delta)$. Buyers purchase trees only if the expected value of a purchased tree is equal to the value of the foregone fruit, $p\lambda$. More formally,

Definition 5 *A pooling equilibrium with adverse selection is a triple of functions $v_h : [\underline{\delta}, \bar{\delta}] \mapsto \mathbb{R}_+$, $v_l : [\underline{\delta}, \bar{\delta}] \mapsto \mathbb{R}_+$, and $\zeta : [\underline{\delta}, \bar{\delta}] \mapsto [0, 1]$, a price $p \in \mathbb{R}_+$, and a number $\lambda \in [1, \beta_h/\beta_l]$ satisfying the following conditions:*

1. *consistency of the value functions: for all $\delta \in [\underline{\delta}, \bar{\delta}]$,*

$$v_h(\delta) = \delta\lambda + \beta_h \bar{v}(\delta) \text{ and } v_l(\delta) = \delta + \max\{p, \beta_l \bar{v}(\delta)\},$$

where $\bar{v}(\delta) \equiv \pi_h v_h(\delta) + \pi_l v_l(\delta)$.

2. *sellers' optimality: for all $\delta \in [\underline{\delta}, \bar{\delta}]$, $\zeta(\delta) = \begin{cases} 1 & \text{if } p > \beta_l \bar{v}(\delta) \\ 0 & \text{if } p < \beta_l \bar{v}(\delta). \end{cases}$*

3. *buyers' optimality: $p\lambda = \beta_h \frac{\int_{\underline{\delta}}^{\bar{\delta}} \zeta(\delta)\bar{v}(\delta)\kappa(\delta)d\delta}{\int_{\underline{\delta}}^{\bar{\delta}} \zeta(\delta)\kappa(\delta)d\delta}$.*

4. *market clearing: $\pi_h \int_{\underline{\delta}}^{\bar{\delta}} \delta\kappa(\delta)d\delta \gtrless \pi_l p \int_{\underline{\delta}}^{\bar{\delta}} \zeta(\delta)\kappa(\delta)d\delta \Rightarrow \lambda = \begin{cases} 1 \\ \beta_h/\beta_l. \end{cases}$*

The definition of a pooling equilibrium consists of four parts. First is the value functions, which state that an individual with a high discount factor values his trees based on the possibility of using the fruit to purchase more trees, while an individual with a low discount factor values them both for their fruit and potentially for their resale value. This immediately implies that trees that produce more fruit are more valuable, $\bar{v}(\delta)$ is increasing.

The second part of the definition of equilibrium states that a seller will sell a tree for sure if the price exceeds the discounted value of the tree and won't sell it if the inequality is reversed. If $p = \beta_l \bar{v}(\delta)$, the sale probability is some arbitrary $\zeta(\delta)$. This implies that all trees below some threshold are sold whenever they are held by a consumer with a low discount factor.

[6] Working directly with a continuum of trees allows us to avoid a technical treatment of tie-breaking.

The third part of the definition states that in equilibrium, buyers pay a fair price for trees, given their valuation of a unit of fruit at λ. The left hand side is the value of the fruit used to purchase a tree, while the right hand side is the expected discounted value of the tree that the buyer obtains.

Finally, the market clearing condition states that if the amount of fruit held by buyers at the beginning of the period exceeds the cost of purchase the trees sold by sellers, then the value of fruit must be equal to its consumption value (since some buyers eat fruit). If it is smaller, then the value of fruit must be so high that sellers must keep some of their worst trees and the value of fruit is driven up to β_h/β_l. Otherwise, λ takes on an intermediate value and buyers do not consume any fruit.

One can prove the existence of equilibrium in this environment. We are more interested in how liquid markets are in this environment. The following proposition is key:

Proposition 6 *In any pooling equilibrium, only trees with $\delta \leq \delta^*$ are sold in the market, where*

$$\delta^* = \frac{\beta_h(1-\bar{\beta})}{\beta_l\big(\lambda - \beta_h(\pi_l + \lambda\pi_h\beta_h)\big)} \frac{\int_{\underline{\delta}}^{\delta^*} \delta\kappa(\delta)d\delta}{\int_{\underline{\delta}}^{\delta^*} \kappa(\delta)d\delta} \tag{12}$$

and $\bar{\beta} = \pi_l\beta_l + \pi_h\beta_h$ is the expected discount factor.

Note that in general there is no guarantee that the pooling equilibrium is unique. Formally, there can be multiple solutions to equation (12). This is because both the left and right hand sides of equation (12) are increasing in δ^* for fixed λ, and the slope of the right hand side may be arbitrarily large, for example when the density κ is large in a neighborhood of δ^*. In that case, there can be an equilibrium with a low price in which sellers are only willing to sell bad trees and buyers pay a low price anticipating that they will purchase only bad trees. There can be another equilibrium in which more trees sell and so buyers are willing to pay more for a tree. Chari, Shourideh and Zetlin-Jones (2010) propose a slightly modified definition of equilibrium which selects the outcome with the highest price, and so we do not view this nonuniqueness as an essential feature of a pooling environment.[7]

Like the competitive equilibrium with adverse selection, some trees are illiquid in the pooling equilibrium, namely those with $\delta > \delta^*$. In fact, the distinction between liquid and illiquid trees is dichotomous in this environment. Trees that are more productive than the critical value are never sold, while trees that are less productive sell whenever they are held by an individual with a low discount factor. Moreover, a seller could always sell an illiquid

[7]In addition, modest regularity conditions on κ like log-concavity are enough to eliminate this source of multiple equilibrium.

asset for the market price p, but he chooses not to do so. In this sense the nature of illiquidity is similar in the two models, although it is more extreme in the pooling environment.

Still, there are important qualitative differences between the two environments. First, in the competitive equilibrium with adverse selection, we predict that higher quality trees will sell at a higher price but take longer to sell. This prediction can be tested empirically. For example, we predict that within a class of securities that look outwardly similar, those that sell for a higher price will take longer to sell but will generate higher dividends on average.[8] In the pooling equilibrium, any two trees within the same class should sell at the same time and should sell as soon as they are offered in the market. The model therefore predicts no correlation between price, time to sell, and dividend.

Second, consider a fire sale. Again, suppose the pooling economy starts from an initial condition in which everyone believes that all trees produce dividend δ_0 and there are lots of buyers, so $\lambda = 1$. A small amount of dispersion in tree quality will not affect the equilibrium, since even a seller with the best tree $\bar{\delta}$ would be willing to sell it for the price of the average tree δ_0. But if the dispersion in tree quality continues to grow, adverse selection will be a problem and so the average quality of trees sold and the equilibrium price will fall. This again looks different than in the competitive equilibrium with adverse selection. In that case, the price of a high quality tree may be higher or lower in the presence of adverse selection than in the initial condition where everyone views the trees as homogeneous. In any case, the market for high quality trees will continue to exist, although it may be thin. In the pooling equilibrium, the price of a high quality tree is unchanged by a small amount of adverse selection, while sellers withdraw it completely from the market when the adverse selection problem is too severe.

Finally, we consider the impact of an asset purchase program. Suppose the "government" stands ready to purchase as many trees as people want to sell at price $\bar{p} > p^*$. In the pooling environment, if there is any more trade in the private market, the private market price has to equal $\bar{p}$. Sellers' indifference condition $\bar{p} = \beta_l \bar{v}(\delta^*)$ then pins down the quality of the marginal tree in the market, while buyers' indifference condition $\bar{p}\lambda = \beta_l \mathbb{E}_{\delta \leq \delta^*} \bar{v}(\delta)$ pins down the average quality. But there is no condition to pin down the amount of trees left in the private market. That is, if there is an equilibrium of the model in which the density of assets with $\delta \leq \delta^*$ is given by $\tilde{\kappa}$ after the intervention, there is another equilibrium in which it is given by $\eta\tilde{\kappa}$ for any $\eta < 1$. In particular, there is always a solution in which arbitrarily few assets are left in the private market.

This might not be a desirable outcome, and so one can imagine the government attempt-

[8] We include the caveat "on average" because dividends may follow a stochastic process in reality. In that case, assets are distinguished based on the expected present value of future dividends.

ing to avoid this by capping the amount of assets it is willing to purchase for $\bar{p}$. In this case, the asset purchase program will be oversubscribed and the government will have to ration its purchases, presumably without knowing the quality of the assets it is buying.[9] It follows that the private market price after the intervention is simply bounded above by $\bar{p}$. If, for example, the government is equally likely to buy any asset that sellers value at less than $\bar{p}$, one can prove that this intervention will not affect the private market price p^* but will instead simply reduce the volume of assets in circulation. If the government is somehow able to screen out the worst assets from the purchase program, perhaps by requiring sellers to hold onto the asset for some time before announcing which assets it will purchase, then the intervention will lower the private market price. In contrast, the asset purchase program appears to be a much more promising intervention if our notion of competitive equilibrium with adverse selection is the relevant one.

[9]This is consistent with the approach in Chiu and Koeppl (2011), who assume that the government chooses both the price it pays for assets and the amount of assets it purchases. In their model, there are only two types of assets and so the government necessarily purchases only bad assets, rendering the indeterminacy we raise here irrelevant.

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Appendix

A Individual's Problem: Details

For any period t, history s^{t-1}, and type $j \in \{1, \dots, J\}$, let $k_{i,j,t}(s^{t-1})$ denote individual i's beginning-of-period t holdings of type j trees. For any period t, history s^t, type $j \in \{1, \dots, J\}$, and set $P \subset \mathbb{R}_+$, let $q_{i,j,t}(P; s^t)$ denote his net purchase in period t of type j trees at a price $p \in P$. The individual chooses a history-contingent sequence for consumption $c_{i,t}(s^t)$ and measures of tree holdings $k_{i,j,t+1}(s^t)$ and net tree purchases $q_{i,j,t}(P; s^t)$ to maximize his expected lifetime utility

$$\sum_{t=0}^{\infty} \sum_{s^t} \left(\prod_{\tau=0}^{t-1} \pi_{s_\tau} \beta_{s_\tau} \right) \pi_{s_t} c_{i,t}(s^t).$$

This simply states that the individual maximizes the expected discounted value of consumption, given the stochastic process for the discount factor. The individual faces a standard budget constraint,

$$\sum_{j=1}^{J} \delta_j k_{i,j,t}(s^{t-1}) = c_{i,t}(s^t) + \int_0^\infty p \left(\sum_{j=1}^{J} q_{i,j,t}(\{p\}; s^t) \right) dp,$$

for all t and s^t. The left hand side is the fruit produced by the trees he owns at the start of period t. The right hand side is consumption plus the net purchase of trees at nonnegative prices p. He also faces a law of motion for his tree holdings,

$$k_{i,j,t+1}(s^t) = k_{i,j,t}(s^{t-1}) + q_{i,j,t}(\mathbb{R}_+; s^t),$$

for all $j \in \{1, \dots, J\}$. This states that the increase in his tree holdings is given by his net purchase of that type of tree. Finally, the individual faces a set of constraints that depends on whether his discount factor is high or low.

If the individual has a high discount factor, $s_t = h$, he is a buyer, which implies $q_{i,j,t}(P; s^t)$ is nonnegative for all $j \in \{1, \dots, J\}$ and $P \subset \mathbb{R}_+$. In addition, he must have enough fruit to purchase trees,

$$\sum_{j=1}^{J} \delta_j k_{i,j,t}(s^{t-1}) \geq \int_0^\infty \max\{\Theta(p), 1\} p \left(\sum_{j=1}^{J} q_{i,j,t}(\{p\}; s^t) \right) dp.$$

If the individual wishes to purchase q trees at a price p and $\Theta(p) > 1$, he will be rationed and so must bring $\Theta(p)pq$ fruit to the market to make this purchase. This constrains his ability to buy trees in markets with excess demand. Together with the budget constraint, this also ensures consumption is nonnegative. Finally, he can only purchase type j trees at a price p if individuals are selling them at that price, that is

$$q_{i,j,t}(P; s^t) = \int_P \gamma_j(p) \left(\sum_{j'=1}^{J} q_{i,j',t}(\{p\}; s^t) \right) dp$$

for all $j \in \{1, \ldots, J\}$ and $P \subset \mathbb{R}_+$. The left hand side is the quantity of type j trees purchased at a price $p \in P$. The integrand on the right hand side is the product of quantity of trees purchased at price p and the share of those trees that are of type j.

If the individual has a low discount factor, $s_t = l$, he is a seller, which implies $q_{i,j,t}(P; s^t)$ is nonpositive for all $j \in \{1, \ldots, J\}$ and $P \subset \mathbb{R}_+$. In addition, he may not try to sell more trees than he owns:

$$k_{i,j,t}(s^{t-1}) \geq -\int_0^\infty \max\{\Theta(p)^{-1}, 1\} q_{i,j,t}(\{p\}; s^t) dp,$$

for all $j \in \{1, \ldots, J\}$. Each tree only sells with probability $\min\{\Theta(p), 1\}$ at price p, so if $\Theta(p) < 1$, an individual must bring $\Theta(p)^{-1}$ trees to the market to sell one of them. Sellers are not restricted from selling trees in the wrong market. Instead, in equilibrium they will be induced not to do so.

Let $\bar{V}^*(\{k_j\})$ be the supremum of the individuals' expected lifetime utility over feasible policies, given initial tree holding vector $\{k_j\}$. We prove in Proposition 1 that the function $\bar{V}^*$ satisfies the following functional equation:

$$\bar{V}(\{k_j\}) = \pi_h V_h(\{k_j\}) + \pi_l V_l(\{k_j\}), \tag{13}$$

where

$$V_h(\{k_j\}) = \max_{\{q_j,k_j'\}} \left(\sum_{j=1}^{J} \delta_j k_j - \int_0^\infty p \left(\sum_{j=1}^{J} q_j(\{p\}) \right) dp + \beta_h \bar{V}(\{k_j'\}) \right) \tag{14}$$

$$\begin{aligned} \text{subject to } & k_j' = k_j + q_j(\mathbb{R}_+) \text{ for all } j \in \{1, \ldots, J\} \\ & \sum_{j=1}^{J} \delta_j k_j \geq \int_0^\infty \max\{\Theta(p), 1\} p \left(\sum_{j=1}^{J} q_j(\{p\}) \right) dp, \\ & q_j(P) = \int_P \gamma_j(p) \left(\sum_{j=1}^{J} q_j(\{p\}) \right) dp \text{ for all } j \in \{1, \ldots, J\} \text{ and } P \subset \mathbb{R}_+ \\ & q_j(P) \geq 0 \text{ for all } j \in \{1, \ldots, J\} \text{ and } P \subset \mathbb{R}_+, \end{aligned}$$

and

$$V_l(\{k_j\}) = \max_{\{q_j,k_j'\}} \left(\sum_{j=1}^{J} \delta_j k_j - \int_0^\infty p \left(\sum_{j=1}^{J} q_j(\{p\}) \right) dp + \beta_l \bar{V}(\{k_j'\}) \right) \tag{15}$$

$$\begin{aligned} \text{subject to } & k_j' = k_j + q_j(\mathbb{R}_+) \text{ for all } j \in \{1, \ldots, J\} \\ & k_j \geq -\int_0^\infty \max\{\Theta(p)^{-1}, 1\} q_j(\{p\}) dp \text{ for all } j \in \{1, \ldots, J\}, \\ & q_j(P) \leq 0 \text{ for all } j \in \{1, \ldots, J\} \text{ and } P \subset \mathbb{R}_+, \end{aligned}$$

We now prove Proposition 1 working with the recursive version of the individuals' problem.

Let $\bar{\Theta}(p) \equiv \max\{\Theta(p), 1\}$ and $\underline{\Theta}(p) = \min\{\Theta(p), 1\}$. Fix Θ and Γ and take any positive-valued numbers $\{v_{s,j}\}$ and λ that solve the Bellman equations (1), (3), and (4) for $s = l, h$. Let p_h be an optimal price for buying trees,

$$p_h \in \arg\max_p \left(\bar{\Theta}(p)^{-1} \left(\frac{\beta_h \sum_{j=1}^{J} \gamma_j(p) \bar{v}_j}{p} - 1 \right) \right).$$

Similarly let $p_{l,j}$ be an optimal price for selling type j trees,

$$p_{l,j} = \arg\max_p \underline{\Theta}(p) \big(p - \beta_l \bar{v}_j\big)$$

for all δ. We seek to prove that $\bar{V}^*(\{k_j\}) \equiv \sum_{j=1}^{J} \bar{v}_j k_j$ where $\bar{v}_j = \pi_h v_{h,j} + \pi_l v_{l,j}$.

If $\lambda = 1$, equations (1) and (3) imply

$$\bar{v}_j = \pi_h \big(\delta_j + \beta_h \bar{v}_j\big) + \pi_l \big(\delta_j + \underline{\Theta}(p_{l,j}) p_{l,j} + (1 - \underline{\Theta}(p_{l,j})) \beta_l \bar{v}_j\big).$$

for all δ. Equivalently,

$$\bar{v}_j = \frac{\delta_j + \pi_l \underline{\Theta}(p_{l,j}) p_{l,j}}{1 - \pi_h \beta_h - \pi_l \beta_l (1 - \underline{\Theta}(p_{l,j}))} > 0.$$

Alternatively, if $\lambda > 1$, the same equations imply

$$\bar{v}_j = \pi_h \left(\delta_j \left(\left(1 - \bar{\Theta}(p_h)^{-1}\right) + \bar{\Theta}(p_h)^{-1} \frac{\beta_h \sum_{j'=1}^{J} \gamma_{j'}(p_h) \bar{v}_{j'}}{p_h} \right) + \beta_h \bar{v}_j \right) \\ + \pi_l \big(\delta_j + \underline{\Theta}(p_{l,j}) p_{l,j} + (1 - \underline{\Theta}(p_{l,j})) \beta_l \bar{v}_j \big)$$

for all δ. Since $v_{l,j}$ and $v_{h,j}$ are positive by assumption so is $\bar{v}_j$, and equivalently we can write

$$\bar{v}_j \left(1 - \pi_h \beta_h - \pi_l \beta_l (1 - \underline{\Theta}(p_{l,j})) - \pi_h \beta_h \bar{\Theta}(p_h)^{-1} \frac{\delta_j \sum_{j'=1}^{J} \gamma_{j'}(p_h) \bar{v}_{j'}}{p_h \bar{v}_j} \right) \\ = \pi_h \delta_j \left(1 - \bar{\Theta}(p_h)^{-1}\right) + \pi_l \big(\delta_j + \underline{\Theta}(p_{l,j}) p_{l,j} \big).$$

The right hand side of this expression is positive for all j. Once again since $\bar{v}_j > 0$, with $\lambda > 1$, this holds if and only if

$$1 - \pi_h \beta_h - \pi_l \beta_l (1 - \underline{\Theta}(p_{l,j})) > \pi_h \beta_h \bar{\Theta}(p_h)^{-1} \frac{\delta_j \sum_{j'=1}^{J} \gamma_{j'}(p_h) \bar{v}_{j'}}{p_h \bar{v}_j}. \tag{16}$$

If this restriction fails at any prices p_h and $p_{l,j}$, it is possible for an individual to obtain unbounded expected utility by buying and selling trees at the appropriate prices. We are interested in cases in which it is satisfied.

Next, let $\bar{V}(\{k_j\}) = \sum_{j=1}^{J} \bar{v}_j k_j$ and $V_s(\{k_j\}) \equiv \sum_{j=1}^{J} v_{s,j} k_j$ for $s = l, h$. It is straightforward to prove that $\bar{V}$ and $\bar{V}_s$ solve equations (13), (14), and (15) and that the same policy is optimal. (Include proof?)

Finally, we adapt Theorem 4.3 from Werning (2009), which states the following: suppose $\bar{V}(k)$ for all k satisfies the recursive equations (13), (14), and (15) and there exists a plan that is optimal given this value function which gives rise to a sequence of tree holdings $\{k^*_{i,j,t}(s^{t-1})\}$ satisfying

$$\lim_{t \to \infty} \sum_{s^t} \left(\prod_{\tau=0}^{t-1} \pi_{s_\tau} \beta_{s_\tau} \right) \bar{V}(\{k^*_{i,j,t}(s^{t-1})\}) = 0. \tag{17}$$

Then, $\bar{V}^* = \bar{V}$.

If $\lambda = 1$, an optimal plan is to sell type j trees at price $p_{l,j}$ when impatient and not to purchase trees when patient. This gives rise to a non-increasing sequence for tree holdings. Given the linearity of $\bar{V}$, condition (17) holds trivially.

If $\lambda > 1$, it is still optimal to sell type j trees at price $p_{l,j}$ when impatient, but patient individuals purchase trees at price p_h and do not consume. Thus

$$\begin{aligned} k'_{h,j} &= k_j + \bar{\Theta}(p_h)^{-1}\gamma_j(p_h)\frac{\sum_{j'=1}^{J}\delta_{j'}k_{j'}}{p_h} \\ k'_{l,j} &= \big(1 - \underline{\Theta}(p_{l,j})\big)k_j. \end{aligned}$$

Using linearity of the value function, the expected discounted value next period of an individual with tree holdings $\{k_j\}$ this period is

$$\begin{aligned} &\sum_{j=1}^{J}\bar{v}_j\big(\pi_h\beta_h k'_{h,j} + \pi_l\beta_l k'_{l,j}\big) \\ &\quad = \sum_{j=1}^{J}\bar{v}_j\left(\pi_h\beta_h\left(k_j + \bar{\Theta}(p_h)^{-1}\gamma_j(p_h)\frac{\sum_{j'=1}^{J}\delta_{j'}k_{j'}}{p_h}\right) + \pi_l\beta_l\big(1-\underline{\Theta}(p_{l,j})\big)k_j\right) \\ &\qquad = \sum_{j=1}^{J}\bar{v}_j k_j\left(\pi_h\beta_h + \pi_l\beta_l\big(1-\underline{\Theta}(p_{l,j})\big) + \pi_h\beta_h\bar{\Theta}(p_h)^{-1}\frac{\delta_j\sum_{j'=1}^{J}\gamma_{j'}(p_h)\bar{v}_{j'}}{p_h\bar{v}_j}\right), \end{aligned}$$

where the second equality simply rearranges terms in the summation. Equation (16) implies that each term of this sum is strictly smaller than $\bar{v}_j k_j$. This implies that there exists an $\eta < 1$ such that

$$\eta > \frac{\sum_{j=1}^{J}\bar{v}_j\big(\pi_h\beta_h k'_{h,j} + \pi_l\beta_l k_{l,j}\big)}{\sum_{j=1}^{J}\bar{v}_j k_j} = \frac{\pi_h\beta_h\bar{V}(\{k'_{h,j}\}) + \pi_l\beta_l\bar{V}(\{k'_{l,j}\})}{\bar{V}(\{k_j\})},$$

and so condition (17) holds.

B Omitted Proofs

Proof of Lemma 1. Consider problem (P_1). Given that there is no $j' < 1$, the only constraint is (5). If such a constraint were slack, we could increase p and hence raise the value of the objective function, which ensures the constraint binds. Eliminating the price by substituting the binding constraint into the objective function gives

$$v_{l,1} = \delta_1 + \max_{\theta} \left(\min\{\theta, 1\} \frac{\beta_h \min\{\theta^{-1}, 1\}}{\lambda - 1 + \min\{\theta^{-1}, 1\}} + (1 - \min\{\theta, 1\})\beta_l \right) \bar{v}_1.$$

If $\lambda = 1$, any $\theta_1 \geq 1$ attains the maximum. If $\lambda = \beta_h/\beta_l$, any $\theta_1 \in [0, 1]$ attains the maximum. For intermediate values of λ, the unique maximizer is $\theta_1 = 1$. Substituting back into the original problem gives $v_{l,1} = \delta_1 + p_1$ and $p_1 = \beta_h \bar{v}_1/\lambda$, establishing the result for $j = 1$.

For $j \geq 2$ we proceed by induction. Assume for all $j' \in \{2, \ldots, j-1\}$, we have established the characterization of $p_{j'}$, $\theta_{j'}$, $v_{l,j'}$ and $\bar{v}_{j'}$ in the statement of the lemma. We first prove that $\bar{v}_j > \bar{v}_{j-1}$. To do this, consider the policy (θ_{j-1}, p_{j-1}). If this solved problem (P_j), combining the objective function and the definition of $\bar{v}_j$ gives

$$\begin{aligned}
\bar{v}_j &= \frac{\delta_j(\pi_h\lambda + \pi_l) + \pi_l \min\{\theta_{j-1}, 1\} p_{j-1}}{1 - \pi_h\beta_h - \pi_l\beta_l(1 - \min\{\theta_{j-1}, 1\})} \\
&> \frac{\delta_{j-1}(\pi_h\lambda + \pi_l) + \pi_l \min\{\theta_{j-1}, 1\} p_{j-1}}{1 - \pi_h\beta_h - \pi_l\beta_l(1 - \min\{\theta_{j-1}, 1\})} = \bar{v}_{j-1}.
\end{aligned}$$

The inequality uses the fact that the denominator is positive together with $\delta_j > \delta_{j-1}$; and the last equality comes from the objective function and the definition of $\bar{v}_{j-1}$ in problem (P_{j-1}). Since the proposed policy satisfies all of the constraints in problem (P_{j-1}) and $\bar{v}_j > \bar{v}_{j-1}$, it also satisfies all the constraints in problem (P_j). The optimal policy must deliver a weakly higher value, proving $\bar{v}_j > \bar{v}_{j-1}$.

Next we prove that at any solution to problem (P_j) the constraint (5) is binding. If there were an optimal policy (θ, p) such that it was slack, consider a small increase in p to $p' > p$ and a reduction in θ to $\theta' < \theta$ so that $\min\{\theta, 1\}(p - \beta_l\bar{v}_{j-1}) = \min\{\theta', 1\}(p' - \beta_l\bar{v}_{j-1})$ while constraint (5) is still satisfied. Now suppose for some $j' \neq j - 1$, $\min\{\theta, 1\}(p - \beta_l\bar{v}_{j'}) < \min\{\theta', 1\}(p' - \beta_l\bar{v}_{j'})$. Subtracting the inequality from the preceding equation gives

$$\big(\min\{\theta, 1\} - \min\{\theta', 1\}\big)(\bar{v}_{j'} - \bar{v}_{j-1}) > 0.$$

Given that $\theta' < \theta$, the above inequality yields $\bar{v}_{j'} > \bar{v}_{j-1}$ and hence $j' \geq j$. This implies that the change in policy does not tighten the constraints (6) for $j' < j$, while it raises the value of the objective function in problem (P_j), a contradiction. Therefore constraint (5) must bind at the optimum.

We now show that the binding constraint (5) implies that $\theta_j \leq 1$ for all $j \geq 2$. By contradiction, assume that the solution to problem (P_j) is some (θ, p) with $\theta > 1$. In this case, the objective function reduces to $v_{l,j} = \delta_j + p$, while the constraint (6) for $j' = 1$

imposes $v_{l,1} \geq \delta_1 + p$. Since we have shown that $v_{l,1} = \delta_1 + p_1$, this implies $p \leq p_1$. Moreover, $\bar{v}_j > \bar{v}_1$ implies $\beta_h \bar{v}_j/\lambda > \beta_h \bar{v}_1/\lambda = p_1$ and hence $\beta_h \bar{v}_j/p > \lambda$. Now a change to the policy $(1, p)$ relaxes the constraint (5) without affecting any other piece of the problem (P_j) and is therefore weakly optimal. But this cannot be optimal because (5) is slack, a contradiction. This proves that $\theta_j \leq 1$ for all $j \geq 2$ and hence, using the binding constraint (5), $p_j = \beta_h \bar{v}_j/\lambda$.

Next, we prove that if $\lambda < \beta_h/\beta_l$, the constraint (6) is binding at $j' = j-1$. We break our proof into two parts. First, consider $j = 2$ and, to find a contradiction, assume that there is a solution (θ, p) to problem (P_2) such that constraint (6) is slack for $j' = 1$. Then problem (P_2) is equivalent to problem (P_1) except for the value of the dividend $\delta_2 > \delta_1$. Following the same argument used for problem (P_1), we can show that $\theta_2 \geq 1$ and so constraint (6) reduces to $v_{l,1} \geq \delta_1 + p_2$. But since $p_1 = \beta_h \bar{v}_1/\lambda < p_2 = \beta_h \bar{v}_2/\lambda$, this contradicts $v_{l,1} = \delta_1 + p_1$. Constraint (6) must bind when $j = 2$.

Next consider $j > 2$ and again assume by contradiction that there is a solution (θ, p) to problem (P_j) such that constraint (6) is slack for $j' = j-1$. Then problem (P_j) is equivalent to problem (P_{j-1}) except in the value of the dividend δ. Since constraint (6) is binding in the solution to problem (P_{j-1}) and $\theta_{j-1} \leq 1$, we have

$$v_{l,j-2} = \delta_{j-2} + \theta_{j-1} p_{j-1} + (1 - \theta_{j-1})\beta_l \bar{v}_{j-2} = \delta_{j-2} + \theta p + (1-\theta)\beta_l \bar{v}_{j-2},$$

and hence

$$\theta_{j-1}(p_{j-1} - \beta_l \bar{v}_{j-2}) = \theta(p - \beta_l \bar{v}_{j-2}). \tag{18}$$

Since $p = \beta_h \bar{v}_j/\lambda$ and $p_{j-1} = \beta_h \bar{v}_{j-1}/\lambda$, $p - \beta_l \bar{v}_{j-2} > p_{j-1} - \beta_l \bar{v}_{j-2} > 0$ and so $\theta_{j-1} > \theta > 0$. But now combine equation (18) with $\theta_{j-1} > \theta$ and $\bar{v}_{j-1} > \bar{v}_{j-2}$ to get

$$\theta_{j-1}(p_{j-1} - \beta_l \bar{v}_{j-1}) < \theta(p - \beta_l \bar{v}_{j-1}).$$

This implies that constraint (6) for $j' = j-1$ is violated, a contradiction. This proves that constraint (6) must bind whenever $\lambda < \beta_j/\beta_l$ and establishes all the equations in the statement of the lemma.

Alternatively, suppose $\lambda = \beta_h/\beta_l$. Since $p_j = \beta_h \bar{v}_j/\lambda = \beta_l \bar{v}_j$, the objective function in problem (P_j) reduces to $v_{l,j} = \delta_j + \beta_l \bar{v}_j$, while constraint (6) imposes

$$v_{l,j'} = \delta_{j'} + \beta_l \bar{v}_{j'} \geq \delta_{j'} + \beta_l \big(\theta \bar{v}_j + (1-\theta)\bar{v}_{j'}\big)$$

for all $j' < j$. Since $\bar{v}_j > \bar{v}_{j'}$, this implies $\theta = 0$ in the solution to the problem. It is easy to verify that this is implied by the equations in the statement of the lemma.

Finally, we need to prove that there is a unique value of $\bar{v}_j > \bar{v}_{j-1}$ that solves the four equations in the statement of the lemma. Combining them we obtain

$$(1-\pi_h\beta_h-\pi_l\beta_l)\bar{v}_j=\delta_j(\pi_l+\lambda\pi_h)+\pi_l\min\{\theta_{j-1},1\}\frac{(\beta_h-\beta_l\lambda)^2\bar{v}_{j-1}\bar{v}_j}{(\beta_h\bar{v}_j-\beta_l\lambda\bar{v}_{j-1})\lambda}. \tag{19}$$

If $\lambda=\beta_h/\beta_l$, the last term is zero and so this pins down $\bar{v}_j$ uniquely. Otherwise we prove that there is a unique solution to equation (19) with $\bar{v}_j>\bar{v}_{j-1}$. In particular, the left hand side is a linearly increasing function of $\bar{v}_j$, while the right hand side is an increasing, concave function, and so there are at most two solutions to the equation. As $\bar{v}_j\to\infty$, the left hand side exceeds the right hand side, and so we simply need to prove that as $\bar{v}_j\to\bar{v}_{j-1}$, the right hand side exceeds the left hand side.

First assume $j=2$ so $\theta_{j-1}=\theta_1\geq 1$. Then we seek to prove that

$$(1-\pi_h\beta_h-\pi_l\beta_l)\bar{v}_1<\delta_2(\pi_l+\lambda\pi_h)+\pi_l\frac{(\beta_h-\beta_l\lambda)\bar{v}_1}{\lambda}.$$

Since $\bar{v}_1=(\delta_1\lambda(\pi_l+\lambda\pi_h))/(\lambda-\beta_h(\pi_l+\lambda\pi_h))$ and $\delta_1<\delta_2$, we can confirm this directly. Next take $j\geq 3$. In this case, in the limit with $\bar{v}_j\to\bar{v}_{j-1}$, the right hand side of (19) converges to

$$\delta_j(\pi_l+\lambda\pi_h)+\pi_l\theta_{j-1}\frac{(\beta_h-\beta_l\lambda)\bar{v}_{j-1}}{\lambda}>\delta_{j-1}(\pi_l+\lambda\pi_h)+\pi_l\min\{\theta_{j-2},1\}\frac{(\beta_h-\beta_l\lambda)^2\bar{v}_{j-2}\bar{v}_{j-1}}{(\beta_h\bar{v}_{j-1}-\beta_l\lambda\bar{v}_{j-2})\lambda},$$

where the inequality uses the indifference condition

$$\min\{\theta_{j-2},1\}(p_{j-2}-\beta_l\bar{v}_{j-2})=\theta_{j-1}(p_{j-1}-\beta_l\bar{v}_{j-2})$$

and the assumption $\delta_{j-1}<\delta_j$. The right hand side of the inequality is the same as the right hand side of equation (19) for type $j-1$. The desired inequality then follows by comparing the left hand side of the inequality to the left hand side of equation (19) for type $j-1$. This completes the proof. ■

Proof of Proposition 2.

We first prove that the solution to problem (P) describes a partial equilibrium and then prove that there is no other equilibrium.

Existence. As described in the statement of the proposition, we look for a partial equilibrium where $\mathbb{P}=\{p_j\}$, $\Theta(p_j)=\theta_j$, $\gamma_j(p_j)=1$, $dF(p_j)=K_j/\sum_{j'}K_{j'}$, and $v_{s,j}$ solves problem (P_j). Also for notational convenience define $p_{J+1}=\infty$. To complete the characterization, we define Θ and Γ on their full support $\mathbb{R}_+$. For $p<p_1$, $\Theta(p)=\infty$ and $\Gamma(p)$ can be chosen

arbitrarily, for example $\gamma_1(p) = 1$. For $j \in \{1, \dots, J\}$ and $p \in (p_j, p_{j+1})$, $\gamma_j(p) = 1$ and $\Theta(p)$ satisfies sellers' indifference condition $v_{l,j} = \delta_j + \min\{\Theta(p), 1\}p + \big(1 - \min\{\Theta(p), 1\}\big)\beta_l\bar{v}_j$; equivalently, $\min\{\Theta(p_j), 1\}(p_j - \beta_l\bar{v}_j) = \min\{\Theta(p), 1\}(p - \beta_l\bar{v}_j)$. To prove that this is a partial equilibrium, we need to verify that the five equilibrium conditions hold.

To show that the first third and fourth equilibrium conditions—Buyers' Optimality and Active Markets—are satisfied, it is enough to prove that the prices $\{p_j\}$ solve the optimization problem in equation (4). Lemma 1 implies that $p_j = \beta_h\bar{v}_j/\lambda$ for all λ and j; and $\Theta(p_j) \leq 1$ if $\lambda > 1$. Together these conditions imply that any price p_j achieves the maximum in this optimization problem. For any price $p \in (p_j, p_{j-1})$, $\gamma_j(p) = 1$ by construction, and so the right hand side of equation (4) is smaller than when evaluate at p_j. Moreover, for any $p < p_1$, $\Theta(p) = \infty$ and so the right hand side is $1 \leq \lambda$.

Next we prove that $\min\{\Theta(p_j), 1\}(p_j - \beta_l\bar{v}_j) \geq \min\{\Theta(p), 1\}(p - \beta_l\bar{v}_j)$ for all j and p, with equality if $p \in [p_j, p_{j+1})$. The first and second equilibrium conditions—Sellers's Optimality and Rational Beliefs— follow immediately from this. The equality holds by construction. Let us now focus on the inequalities.

First take any $j' \in \{2, \dots, J\}$, $j < j'$, and $p \in [p_{j'}, p_{j'+1})$. By the construction of Θ,

$$\min\{\Theta(p_{j'}), 1\}(p_{j'} - \beta_l\bar{v}_{j'}) = \min\{\Theta(p), 1\}(p - \beta_l\bar{v}_{j'}).$$

Then $p_{j'} \leq p$ implies that $\min\{\Theta(p_{j'}), 1\} \geq \min\{\Theta(p), 1\}$. Since $j < j'$, Lemma 1 implies that $\bar{v}_{j'} > \bar{v}_j$ and so $\min\{\Theta(p_{j'}), 1\}(\bar{v}_{j'} - \bar{v}_j) \geq \min\{\Theta(p), 1\}(\bar{v}_{j'} - \bar{v}_j)$. Adding this to the previous equation gives $\min\{\Theta(p_{j'}), 1\}(p_{j'} - \beta_l\bar{v}_j) \geq \min\{\Theta(p), 1\}(p - \beta_l\bar{v}_j)$. Also condition (6) in problem $(P_{j'})$ implies $\min\{\Theta(p_j), 1\}(p_j - \beta_l\bar{v}_j) \geq \min\{\Theta(p_{j'}), 1\}(p_{j'} - \beta_l\bar{v}_j)$. Combining the last two inequalities gives $\min\{\Theta(p_j), 1\}(p_j - \beta_l\bar{v}_j) \geq \min\{\Theta(p), 1\}(p - \beta_l\bar{v}_j)$ for all $p \in [p_{j'}, p_{j'+1})$ and $j < j'$.

Similarly, take any $j' \in \{1, \dots, J-1\}$, $j > j'$, and $p \in [p_{j'}, p_{j'+1})$. The construction of Θ implies $\min\{\Theta(p_{j'}), 1\}(p_{j'} - \beta_l\bar{v}_{j'}) = \min\{\Theta(p), 1\}(p - \beta_l\bar{v}_{j'})$, while Lemma 1 together with $\Theta(p_j) = \theta_j$ implies $\min\{\Theta(p_{j'}), 1\}(p_{j'} - \beta_l\bar{v}_{j'}) = \min\{\Theta(p_{j'+1}), 1\}(p_{j'+1} - \beta_l\bar{v}_{j'})$. The two equalities together imply

$$\min\{\Theta(p_{j'+1}), 1\}(p_{j'+1} - \beta_l\bar{v}_{j'}) = \min\{\Theta(p), 1\}(p - \beta_l\bar{v}_{j'})$$

Then $p_{j'+1} > p$ implies $\min\{\Theta(p_{j'+1}), 1\} \leq \min\{\Theta(p), 1\}$. Since $j > j'$, Lemma 1 implies that $\bar{v}_j > \bar{v}_{j'}$ and so $\min\{\Theta(p_{j'+1}), 1\}(\bar{v}_{j'} - \bar{v}_j) \geq \min\{\Theta(p), 1\}(\bar{v}_{j'} - \bar{v}_j)$. Adding this to the previous equation gives $\min\{\Theta(p_{j'+1}), 1\}(p_{j'+1} - \beta_l\bar{v}_j) \geq \min\{\Theta(p), 1\}(p - \beta_l\bar{v}_j)$. Also, since $(\Theta(p_{j'+1}), p_{j'+1})$ is a feasible policy in problem (P_j), $\min\{\Theta(p_j), 1\}$ $(p_j - \beta_l\bar{v}_j) \geq \min\{\Theta(p_{j'+1}), 1\}(p_{j'+1} - \beta_l\bar{v}_j)$. Combining inequalities gives $\min\{\Theta(p_j), 1\}(p_j - \beta_l\bar{v}_j) \geq$

$\min\{\Theta(p), 1\}(p - \beta_l \bar{v}_j)$ for all $p \in [p_{j'}, p_{j'+1})$ and $j > j'$.

Finally, consider $p < p_1$. Since $\Theta(p) = \infty$, $\min\{\Theta(p), 1\}(p - \beta_l \bar{v}_j) = p - \beta_l \bar{v}_j < p_1 - \beta_l \bar{v}_j \leq \min\{\Theta(p_1), 1\}(p_1 - \beta_l \bar{v}_j)$, where the first inequality uses $p < p_1$ and the second uses the fact that $\Theta(p_1) < 1$ only if $\lambda = \beta_h/\beta_l$; but in this case, $p_1 = \beta_l \bar{v}_1 \leq \beta_l \bar{v}_j$. Since we have already proved that $\min\{\Theta(p_1), 1\}(p_1 - \beta_l \bar{v}_j) \leq \min\{\Theta(p_j), 1\}(p_j - \beta_l \bar{v}_j)$, this establishes the inequality for $p < p_1$.

The last piece of the definition of equilibrium is Consistency of Supplies with Beliefs. This holds by the construction of the distribution function F in the statement of the Proposition.

Uniqueness. Now take any partial equilibrium $\{v_h, v_l, \Theta, \Gamma, \mathbb{P}, F\}$. We first claim that $\bar{v}$ is increasing in j. Take $j > j'$ and let $p_{j'}$ denote the price offered by j'. Type j Sellers' Optimality implies

$$v_{l,j} \geq \delta_j + \min\{\Theta(p_{j'}), 1\}p_{j'} + (1 - \min\{\Theta(p_{j'}), 1\})\beta_l \bar{v}_j,$$

and so combining with type j Buyers' Optimality, equation (3), and solving for $\bar{v}_j$ gives

$$\bar{v}_j \geq \frac{\delta_j(\pi_l + \pi_h\lambda) + \pi_l \min\{\Theta(p_{j'}), 1\}p_{j'}}{\pi_l(1 - \min\{\Theta(p_{j'}), 1\})\beta_l + \pi_h\beta_h} > \frac{\delta_{j'}(\pi_l + \pi_h\lambda) + \pi_l \min\{\Theta(p_{j'}), 1\}p_{j'}}{\pi_l(1 - \min\{\Theta(p_{j'}), 1\})\beta_l + \pi_h\beta_h} = \bar{v}_{j'},$$

where the second inequality uses $\delta_j > \delta_{j'}$ and the equality solves the same equations for $\bar{v}_{j'}$.

Consistency of Supplies with Beliefs implies that for each $j \in \{1, \ldots, J\}$, there exists a price $p_j \in \mathbb{P}$ with $\gamma_j(p_j) > 0$.

Now in the remainder of the proof, assume also that $\theta_j \equiv \Theta(p_j) > 0$. First we prove that the constraint $\lambda \leq \min\{\theta_j^{-1}, 1\}\beta_h \bar{v}_j/p_j + (1 - \min\{\theta_j^{-1}, 1\})$ is satisfied. Second we prove that the constraint $v_{l,j'} \geq \delta_{j'} + \min\{\theta_j, 1\}p_j + (1 - \min\{\theta_j, 1\})\beta_l \bar{v}_{j'})$ is satisfied for all $j' < j$. Third we prove that the pair (θ_j, p_j) delivers value $v_{l,j}$ to sellers of type j trees. Fourth we prove that (θ_j, p_j) solves (P_j).

Step 1. To derive a contradiction, assume $\lambda > \min\{\theta_j^{-1}, 1\}\beta_h \bar{v}_j/p_j + 1 - \min\{\theta_j^{-1}, 1\}$. Active Markets implies that the expected value of a unit of fruit to a buyer who pays p_j must equal λ and so there must be a j' with $\gamma_{j'}(p_j) > 0$ and $\lambda < \min\{\theta_j^{-1}, 1\}\beta_h \bar{v}_{j'}/p_j + 1 - \min\{\theta_j^{-1}, 1\}$. If $\theta_j = \infty$, $\min\{\theta_j^{-1}, 1\}\beta_h \bar{v}_{j'}/p_j + 1 - \min\{\theta_j^{-1}, 1\} = 1 \leq \lambda$, which is impossible; therefore $\theta_j < \infty$. Then Rational Beliefs implies p_j is an optimal price for type j' sellers and so for all p' and $\theta' \equiv \Theta(p')$, $\min\{\theta_j, 1\}(p_j - \beta_l \bar{v}_{j'}) \geq \min\{\theta', 1\}(p' - \beta_l \bar{v}_{j'})$. Since $\theta_j > 0$, $\min\{\theta_j, 1\}(p' - \beta_l \bar{v}_{j'}) > \min\{\theta_j, 1\}(p_j - \beta_l \bar{v}_{j'})$ for all $p' > p_j$, and so the two inequalities imply $\min\{\theta_j, 1\} > \min\{\theta', 1\}$.

Now take any $j'' < j'$, so $\bar{v}_{j''} < \bar{v}_{j'}$. Then since $\min\{\theta_j, 1\}(p_j - \beta_l \bar{v}_{j'}) \geq \min\{\theta', 1\}(p' -$

$\beta_l\bar{v}_{j'}$), $\min\{\theta_j,1\} > \min\{\theta',1\}$, and $\bar{v}_{j''} < \bar{v}_{j'}$,

$$\min\{\theta_j,1\}(p_j - \beta_l\bar{v}_{j''}) > \min\{\theta',1\}(p' - \beta_l\bar{v}_{j''}).$$

Type j'' Sellers' Optimality condition implies $\bar{v}_{j''} \geq \delta_{j''} + \min\{\theta_j,1\}p_j + (1-\min\{\theta_j,1\})\beta_l\bar{v}_{j''}$ and so the previous inequality gives $\bar{v}_{j''} > \delta_{j''} + \min\{\theta',1\}p' + (1-\min\{\theta',1\})\beta_l\bar{v}_{j''}$. Rational beliefs implies $\gamma_{j''}(p') = 0$. That is, any $p' > p_j$ attracts only type j' sellers or higher and so delivers value at least equal to $\min\{\theta'^{-1},1\}\beta_h\bar{v}_{j'}/p' + (1-\min\{\theta'^{-1},1\})$ to buyers. For p' sufficiently close to p_j, this exceeds λ, contradicting buyers' optimality.

Step 2. Sellers' Optimality implies $v_{l,j'} \geq \delta_{j'} + \min\{\theta_j,1\}p_j + (1-\min\{\theta_j,1\})\beta_l\bar{v}_{j'})$ for all j', p_j, and $\theta_j = \Theta(p_j)$.

Step 3. Rational Beliefs implies $v_{l,j} = \delta_j + \min\{\theta_j,1\}p_j + (1-\min\{\theta_j,1\})\beta_l\bar{v}_{j'})$ for all j, p_j, and $\theta_j = \Theta(p_j) < \infty$ with $\gamma_j(p_j) > 0$.

Step 4. Suppose there is a policy (θ,p) that satisfies the constraints of problem (P_j) and delivers a higher payoff. That is,

$$\begin{gathered}
v_{l,j} < \delta_j + \min\{\theta,1\}p + (1-\min\{\theta,1\})\beta_l\bar{v}_j \\
\lambda \leq \min\{\theta^{-1},1\}\beta_h\bar{v}_j/p + 1 - \min\{\theta^{-1},1\} \\
v_{l,j'} \geq \delta_{j'} + \min\{\theta,1\}p + (1-\min\{\theta,1\})\beta_l\bar{v}_{j'} \text{ for all } j' < j.
\end{gathered}$$

If these inequalities hold with $\theta > 1$, then the same set of inequalities holds with $\theta = 1$, and so we may assume $\theta \leq 1$ without loss of generality. Choose $p' < p$ such that

$$v_{l,j} < \delta_j + \theta p' + (1-\theta)\beta_l\bar{v}_j \tag{20}$$

$$\lambda < \beta_h\bar{v}_j/p' \tag{21}$$

$$v_{l,j'} > \delta_{j'} + \theta p' + (1-\theta)\beta_l\bar{v}_{j'} \text{ for all } j' < j. \tag{22}$$

The previous inequalities imply that this is always feasible by setting p' close enough to p. Now sellers' optimality implies $v_{l,j} \geq \delta_j + \min\{\Theta(p'),1\}p' + (1-\min\{\Theta(p'),1\})\beta_l\bar{v}_j)$, which, together with inequality (20), implies $\Theta(p') < \theta$. This together with inequality (22) implies that

$$v_{l,j'} > \delta_{j'} + \Theta(p')p' + (1-\Theta(p'))\beta_l\bar{v}_{j'} \text{ for all } j' < j,$$

and so, due to Rational Beliefs, $\gamma_{j'}(p') = 0$ for all $j' < j$. But then, using inequality (21), we obtain

$$\lambda < \frac{\beta_h\bar{v}_j}{p'} \leq \frac{\beta_h\sum_{j'=1}^{J}\gamma_{j'}(p')\bar{v}_{j'}}{p'} = \min\{\Theta(p')^{-1},1\}\frac{\beta_h\sum_{j'=1}^{J}\gamma_{j'}(p')\bar{v}_{j'}}{p'} + \big(1-\min\{\Theta(p')^{-1},1\}\big),$$

where the second inequality uses monotonicity of $\bar{v}_j$ and $\gamma_{j'}(p') = 0$ for $j' < j$; and the last equation uses $\Theta(p') < \theta \leq 1$. This contradicts Buyers' Optimality condition and completes the proof. ■

Proof of Proposition 3. To prove that there exists a unique competitive equilibrium, it is enough to prove that there exists a unique $\lambda \in [1, \beta_h/\beta_l]$ such that the partial equilibrium associated to that λ clears the fruit market.

For given $\lambda \in [1, \beta_h/\beta_l]$, let $x_j(\lambda) \equiv \theta_j(\lambda)p_j(\lambda)$, where $\theta_j(\lambda)$ and $p_j(\lambda)$ are the partial equilibrium sale probability and price for assets of type j. For all $j > 1$ and given $x_{j-1}(\lambda)$, define

$$f_j(x_j, \lambda) \equiv x_j \left[1 - \frac{\beta_l}{\beta_h}\lambda \frac{p_{j-1}(x_{j-1}(\lambda), \lambda)}{p_j(x_j, \lambda)}\right] - x_{j-1}(\lambda)\left[1 - \frac{\beta_l}{\beta_h}\lambda\right],$$

where, with some abuse of notation,

$$p_j(x_j, \lambda) = \frac{\delta_j \beta_h(\pi_l + \lambda\pi_h) + x_j\pi_l[\beta_h - \beta_l\lambda]}{\lambda(1 - \bar{\beta})}. \tag{23}$$

For given $\lambda \in [1, \beta_h/\beta_l]$, Proposition 2 and Lemma 1 ensure that $p_j(x_j(\lambda), \lambda)$ is the equilibrium price for type-j trees with $x_j(\lambda)$ being implicitly defined by $f_j(x_j, \lambda) = 0$ for all $j > 1$. Moreover, for $\lambda \in (1, \beta_h/\beta_l)$

$$x_1(\lambda) = p_1(x_1(\lambda), \lambda) = \frac{\delta_1\beta_h(\pi_l + \lambda\pi_h)}{\lambda - \beta_h(\pi_l + \lambda\pi_h)}. \tag{24}$$

Lemma 1 also implies that $p_j(x_j(\lambda), \lambda) > p_{j-1}(x_{j-1}(\lambda), \lambda)$ for all $j > 1$. From $f_j(x_j, \lambda) = 0$ for all $j > 1$ immediately follows that $x_j(\lambda) < x_{j-1}(\lambda)$ for all $j > 1$.

Next, define $M(\lambda)$ as

$$M(\lambda) \equiv \sum_{j=1}^{J}[\pi_h\delta_j - \pi_l x_j(\lambda)]K_j.$$

Market clearing requires $M(\lambda) = 0$. Now we show that $x_j'(\lambda) < 0$ and hence $M'(\lambda) > 0$ for all $\lambda \in (1, \beta_h/\beta_l)$. For $j = 1$ we can directly calculate

$$x_1'(\lambda) = -\frac{\delta_j\beta_h\pi_l}{[\lambda - \beta_h(\pi_l + \lambda\pi_h)]^2} < 0.$$

For all $j > 1$, given $x_{j-1}'(\lambda) < 0$ we can proceed recursively as follows. Applying the implicit function theorem to $f_j(x_j, \lambda) = 0$, we obtain

$$x_j'(\lambda) = -\frac{\partial f_j(x_j, \lambda)/\partial \lambda}{\partial f_j(x_j, \lambda)/\partial x_j}.$$

First, we can calculate

$$\frac{\partial f_j(x_j, \lambda)}{\partial x_j} = 1 - \frac{\beta_l}{\beta_h}\lambda \frac{p_{j-1}(x_{j-1}(\lambda), \lambda)}{p_j(x_j, \lambda)} + x_j \frac{\beta_l}{\beta_h}\lambda \frac{p_{j-1}(x_{j-1}(\lambda), \lambda)}{p_j(x_j, \lambda)^2} \frac{\partial p_j(x_j, \lambda)}{\partial x_j}.$$

It is easy to show that $\partial f_j(x_j, \lambda)/\partial x_j > 0$ given that $p_j(x_j(\lambda), \lambda) > p_{j-1}(x_{j-1}(\lambda), \lambda)$ and

$$\frac{\partial p_j(x_j, \lambda)}{\partial x_j} = \frac{\pi_l[\beta_h - \beta_l \lambda]}{\lambda(1 - \bar{\beta})} > 0.$$

Second, we can calculate

$$\begin{aligned}\frac{\partial f_j(x_j, \lambda)}{\partial \lambda} =& \frac{\beta_l}{\beta_h}\left[x_{j-1}(\lambda) - x_j \frac{p_{j-1}(x_{j-1}(\lambda), \lambda)}{p_j(x_j, \lambda)}\right] - \left(1 - \frac{\beta_l}{\beta_h}\lambda\right) x_{j-1}'(\lambda) \\ &- \frac{\beta_l}{\beta_h}\lambda \frac{x_j}{p_j(x_j, \lambda)} \frac{\partial p_{j-1}(x_{j-1}(\lambda), \lambda)}{\partial x_{j-1}} x_{j-1}'(\lambda) \\ &- \frac{\beta_l}{\beta_h}\lambda \frac{x_j}{p_j(x_j, \lambda)} \left[\frac{\partial p_{j-1}(x_{j-1}(\lambda), \lambda)}{\partial \lambda} - \frac{p_{j-1}(x_{j-1}(\lambda), \lambda)}{p_j(x_j, \lambda)} \frac{p_j(x_j, \lambda)}{\partial \lambda}\right]\end{aligned}$$

where the first term is positive because $x_j(\lambda) < x_{j-1}(\lambda)$ and $p_j(x_j(\lambda), \lambda) > p_{j-1}(x_{j-1}(\lambda), \lambda)$, the second term is positive because $\lambda \in (1, \beta_h/\beta_l)$ and $x_{j-1}'(\lambda) < 0$, and the third term is positive because of the last inequality together with $\partial p_j(x_j, \lambda)/\partial \lambda > 0$. Finally, to show that the last term is also positive we need to show that the term in square bracket is positive where

$$\frac{\partial p_j(x_j, \lambda)}{\partial x_j} = -\frac{\beta_h \pi_l(\delta_j + x_j)}{\lambda^2(1 - \bar{\beta})}.$$

Using expression (23) for $p_j(x_j, \lambda)$ and $f_j(x_j, \lambda) = 0$ for all j, after some algebra, one can show that this is always the case given that $\lambda \in (1, \beta_h/\beta_l)$. This implies that $x_j'(\lambda) < 0$ for all j and hence $M'(\lambda) > 0$.

Finally, define

$$\underline{\pi} \equiv \frac{\sum_{j=1}^{J} \delta_j K_j}{\sum_{j=1}^{J}[\delta_j + x_j(0)]K_j} \text{ and } \overline{\pi} \equiv \frac{\sum_{j=1}^{J} \delta_j K_j}{\sum_{j=1}^{J}[\delta_j + x_j(\beta_h/\beta_l - 1)]K_j},$$

where $x_1(\lambda)$ is given in equation (24) and $x_j(\lambda)$ solves $f_j(x_j, \lambda) = 0$ for all $j > 1$. It is easy to see that $\underline{\pi} < \overline{\pi}$ given that $x_j'(\lambda) < 0$. Moreover, $M(0) < 0$ iff $\pi_l > \underline{\pi}$ and $M(\beta_h/\beta_l - 1) > 0$ iff

$\pi_l < \overline{\pi}$. Given that $M'(\lambda) > 0$, it follows that if $\pi_l \in (\underline{\pi}, \overline{\pi})$, there exists a unique equilibrium with $\lambda \in (1, \beta_h/\beta_l)$. If instead $\pi_l \leq \underline{\pi}$, then both $M(0)$ and $M(\beta_h/\beta_l - 1)$ are larger than zero, while if $\pi_l \geq \overline{\pi}$, they are both smaller than zero. Lemma 1 implies that $x_1(\lambda) \geq p_1(\lambda)$ if $\lambda = 1$ and $x_1(\lambda) \leq p_1(\lambda)$ if $\lambda = \beta_h/\beta_l$. This implies that if $\pi_l \leq \underline{\pi}$, there exists a unique equilibrium with $\lambda = 1$, where $x_1(0) \geq p_1(0)$ is pinned down by market clearing. If instead $\pi_l \geq \overline{\pi}$, then there exists a unique equilibrium with $\lambda = \beta_h/\beta_l$, where $x_1(0) \leq p_1(0)$ is pinned down by market clearing. This completes the proof. ■

Proof of Proposition 4. We start by establishing that equations (7)–(10) describe an equilibrium. First, in any competitive equilibrium, Sellers' Optimality and Buyers' Optimality imply

$$\begin{aligned}
v_l(\delta) &= \delta + \Theta(P(\delta))P(\delta) + (1 - \Theta(P(\delta)))\beta_l \bar{v}(\delta), \\
v_h(\delta) &= \delta\lambda + \beta_h \bar{v}(\delta).
\end{aligned}$$

Adding π_l times the first equation to π_h times the second and solving for $\bar{v}(\delta)$ gives

$$\bar{v}(\delta) = \frac{\delta(\pi_l + \pi_h\lambda) + \pi_l\Theta(P(\delta))P(\delta)}{1 - \pi_h\beta_h - \pi_l(1 - \Theta(P(\delta)))\beta_l}.$$

Then substitute for $\delta = D(P(\delta))$ using equation (9) and simplify to get $P(\delta) = \beta_h\bar{v}(\delta)/\lambda$, consistent with Lemma 1 in the discrete-type economy. Next, Rational Beliefs implies $P(\delta)$ maximizes $\Theta(p)(p - \beta_l\bar{v}(\delta))$. Using equation (8) for $\Theta(p)$, differentiate this expression to show that it is increasing in p when $p < P(\delta)$ and decreasing when $p > P(\delta)$, where $P(\delta)$ is given by the previous paragraph. The uniquely optimal price for a type δ tree is $P(\delta)$. Thus these prices and this value function satisfy Rational Beliefs. Next, any $p > \underline{p}$ delivers value λ to a buyer by construction, satisfying Active Markets. Consistency of Supply with Beliefs pins down the amount of trees available at each price, $F(P(\delta)) = G(\delta)$ for all δ. With this, the fruit market clearing condition reduces to condition (10).

To show that this is the unique limit of the economy with a finite number of trees, start with the condition that the seller of a type $j \geq 2$ tree must be indifferent about representing it as a type $j+1$ tree. Since $\Theta(p_j) < 1$, $\Theta(p_{j+1})(p_{j+1} - \beta_l\bar{v}_j) = \Theta(p_j)(p_j - \beta_l\bar{v}_j)$, or equivalently

$$\frac{\Theta(p_{j+1}) - \Theta(p_j)}{p_{j+1} - p_j} = -\frac{\Theta(p_j)}{p_{j+1} - \beta_l\bar{v}_j}.$$

Now eliminate $\bar{v}_j$ using the buyer's indifference condition $\bar{v}_j = p_j\lambda/\beta_h$ and take the limit as

$\delta_{j+1} \to \delta_j$, so $p_{j+1} \to p_j$. This gives

$$\Theta'(p_j) = -\frac{\beta_h \Theta(p_j)}{p_j(\beta_h - \beta_l \lambda)}.$$

If $\lambda = \beta_h/\beta_l$, this implies $\Theta(p) = 0$ for all $p > \underline{p}$. Otherwise, solve this differential equation using the terminal condition $\Theta(\underline{p}) = 1$ to get equation (8). The remaining expressions follow immediately from the Bellman equations. ■

Proof of Proposition 5. First, notice that $\bar{\jmath}$ is the highest quality asset that private buyer would not purchase at the government price given that $\delta_{\bar{\jmath}} < \bar{\delta} \leq \delta_{\bar{\jmath}+1}$. The value of any asset $j \in \{1, \ldots, \bar{\jmath}\}$ is determined by the possibility of selling it for $p_j = \bar{p}$ in a perfectly liquid market, $\theta_j = \bar{\theta} \geq 1$. The value, price, and liquidity of higher quality assets $j \in \{\bar{\jmath}+1, \ldots, J\}$ are determined by the standard Bellman equations, asset pricing equation $p_j = \beta_h \bar{v}_j/\lambda$, and incentive constraint $\theta_{j-1}(p_{j-1} - \beta_l \bar{v}_{j-1}) = \theta_j(p_j - \beta_l \bar{v}_{j-1})$, required to keep the owners of type $j-1$ assets from selling them in the type j asset market.

For $j \leq \bar{\jmath}$, $K^p_{j,t} = \pi_h^t K^p_{j,0}$ as individuals with low discount factors sell their low quality trees to the government. We also assume that the fruit that high discount factor individuals get from their high quality trees is just enough to buy the high quality trees held by low discount factor individuals:

$$\pi_h \sum_{j=\bar{\jmath}+1}^{J} \delta_j K^p_{j,0} = \pi_l \sum_{j=\bar{\jmath}+1}^{J} \theta_j p_j K^p_{j,0}.$$

This equation holds because the government sells its stock of high quality trees to the private sector at a rate that absorbs the remaining fruit produced by high discount factor individuals: $K^p_{j,t} = g_t K^p_{j,0}$, where

$$g_t \equiv 1 + \frac{\pi_h(1-\pi_h^t)\sum_{j=1}^{\bar{\jmath}} \delta_j K^p_{j,0}}{(1-\pi_h)\sum_{j=\bar{\jmath}+1}^{J} p_j K^p_{j,0}}.$$

These equations together ensure that high discount factor individuals use all their fruit to buy trees either from low discount factor individuals or from the government. The measure of trees of type j sold by the government to the private individuals is equal to $K^p_{j,t+1} - K^p_{j,t}$. This guarantees that the fruit market clears with a constant value of λ:

$$\pi_h \sum_{j=1}^{J} \delta_j K^p_{j,t} = \pi_l \sum_{j=1}^{J} \theta_j p_j K^p_{j,t} + (g_{t+1} - g_t) \sum_{j=\bar{\jmath}+1}^{J} p_j K^p_{j,0}$$

The left hand side is the fruit held by private individuals with high discount factors, while the right hand side is the fruit needed to purchase trees sold both by the private individuals with low discount factors and by government agents. ■

Proof of Proposition 6. The first part of the definition of a pooling equilibrium implies $\bar{v}(\delta)$ is increasing. Then the second part implies $\zeta(\delta) = 1$ if $\delta < \delta^*$ and $\zeta(\delta) = 0$ if $\delta > \delta^*$, with no restriction on $\zeta(\delta)$ if $\delta = \delta^*$.

Next, for $\delta < \delta^*$, the value functions imply $v_h(\delta) = \delta\lambda + \beta_h \bar{v}(\delta)$ and $v_l(\delta) = \delta + p$. Summing these and solving for $\bar{v}(\delta)$ gives

$$\bar{v}(\delta) = \frac{\delta(\pi_l + \lambda\pi_h) + \pi_l p}{1 - \pi_h \beta_h}.$$

Then $p > \beta_l \bar{v}(\delta)$ if and only if $\delta < \delta^*$ defined by

$$p = \beta_l \frac{\delta^*(\pi_l + \lambda\pi_h)}{1 - \bar{\beta}}. \tag{25}$$

Substituting this back into the previous equation gives

$$\bar{v}(\delta) = \frac{\big((1 - \bar{\beta})\delta + \pi_l \beta_l \delta^*\big)(\pi_l + \lambda\pi_h)}{(1 - \pi_h\beta_h)(1 - \bar{\beta})}. \tag{26}$$

This holds if $\delta \leq \delta^*$; otherwise, the value is reduced by the fact that the asset is never resold.

Next, eliminate p from the buyer's indifference condition, the third part of the definition of equilibrium, using equation (25). Also eliminate $\bar{v}(\delta)$ using equation (26). Solving for δ^* gives

$$\delta^* = \frac{\beta_h(1 - \bar{\beta})}{\beta_l\big(\lambda - \beta_h(\pi_l + \lambda\pi_h)\big)} \frac{\int_{\underline{\delta}}^{\bar{\delta}} \zeta(\delta)\delta\kappa(\delta)d\delta}{\int_{\underline{\delta}}^{\bar{\delta}} \zeta(\delta)\kappa(\delta)d\delta}.$$

Imposing that $\zeta(\delta) = 1$ if $\delta < \delta^*$ and $\zeta(\delta) = 0$ if $\delta > \delta^*$ gives the expression in the statement of the Proposition. ■