# Nominal Debt as a Burden on Monetary Policy*

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**Abstract**

We characterize the optimal sequential choice of monetary policy in economies with either nominal or indexed debt. In a model where nominal debt is the only source of time inconsistency, the (Markov-perfect) sequential optimal policy implies the depletion of the outstanding stock of debt progressively, until the time inconsistency disappears. There is a resulting welfare loss if debt is nominal rather than indexed. We analyze the more general case where monetary policy is time inconsistent even when debt is indexed. In this case, with nominal debt, the sequential optimal policy converges to a time consistent steady state with positive –or negative– debt, depending on the value of the intertemporal elasticity of substitution. Welfare can be higher if debt is nominal rather than indexed and the level of debt is not too high.

Keywords: nominal debt; indexed debt; optimal monetary policy; time consistency; Markov-perfect equilibrium
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## 1 Introduction

Fiscal discipline has often been seen as a precondition to sustain price stability. Such is, for example, the rationale behind the Growth and Stability Pact in Europe. The underlying policy debate shows the concern regarding a time inconsistency problem associated with high levels of nominal debt that could be monetized. In this paper we analyze the implications for the optimal sequential design of monetary policy of public debt being nominal or indexed. We take a simple model where a benevolent government has an incentive to monetize nominal debt, even if this is costly. We characterize the sequentially optimal policy choices with both nominal and indexed debt and assess the relative performance of the two in terms of welfare.

The model is a cash-in-advance production economy where agents start the period with predetermined money balances, used for transactions during the period, as in Svensson (1985). The government's problem is to finance exogenous government expenditures in the least distortionary manner. In this economy, an increase in the price level decreases the real value of outstanding money and nominal debt and therefore reduces the need for distortionary taxation. However, this also induces a fall in present consumption because of the cash-in-advance constraint. As shown by Nicolini (1998), who analyzes the same class of economies, the incentives to inflate, or deflate, depend on preferences and on whether debt is nominal or real.

If debt is indexed, the decision on whether to use the inflation tax, to tax today or tomorrow, hinges on the intertemporal elasticity of substitution. If the elasticity is one then it is equal to the implicit elasticity of the cash-in-advance constraint and the optimal plan is time consistent. However, with nominal debt, there is a reason to monetize the debt, and the optimal policy plan is no longer time consistent. We show that the Markov-perfect sequentially optimal path includes depleting the debt asymptotically, so that the path for the nominal interest rate is decreasing. In this case of unitary elasticity, the fact that debt is nominal rather than indexed introduces a dynamic distortion that lowers welfare unambiguously.

For the general case of non unitary elasticity, optimal taxation principles dictate whether current or future consumption should be taxed more and, as a result, the optimal policy plan is time inconsistent even with indexed debt. In particular, if the intertemporal elasticity of substitution is higher than one – that is, higher than the implicit elasticity of the cash-in-advance constraint – it is efficient to tax more current consumption; along a sequentially optimal path indexed debt is depleted all the way to the first best, where it is negative and large enough in absolute value to finance all expenditures without the need to collect distortionary taxes. If the intertemporal elasticity is, instead, lower than one, future consumption is taxed more and debt increases asymptotically.

With nominal debt, the incentives to inflate when debt is positive can compensate the incentives to deflate when the intertemporal elasticity is lower than one. Similarly, the incentives to deflate when debt is negative can compensate the incentives to inflate when the intertemporal elasticity is higher than one. At the debt level where these conflicting incentives cancel out there is a steady state which is negative (elasticity higher than one) or positive (elasticity lower than one) and the optimal sequential path of nominal debt converges to this steady state. In contrast with the unitary elasticity case, when elasticity is different from one, nominal debt solves – in the long-run – a time inconsistency problem present in the indexed debt case; in particular, if the elasticity is higher than one, there is no need to accumulate as many assets as to achieve the first best, as in the indexed debt case; if the elasticity is lower than one, debt does not increase asymptotically.

A central contribution of this paper is the welfare comparison of the two regimes, nominal or indexed debt. As already mentioned, if the intertemporal elasticity of substitution is one, indexed debt unambiguously dominates nominal debt in terms of welfare. In contrast, if the elasticity is non unitary, the fact that the incentive to monetize the debt can compensate the distortions present with indexed debt can result in nominal debt dominating indexed debt. In particular, as our computations show – when debt is relatively low – nominal debt can be a blessing, rather than a burden, to monetary policy.

Related work includes Calvo (1988), Obstfeld (1997), Nicolini (1998), Ellison and Rankin (2005), Martin (2006), Persson, Persson and Svensson (2006), Reis (2006). Calvo (1988) addressed the question of the relative performance of nominal versus indexed debt, considering a reduced form model with two periods, where nominal debt creates a time inconsistency. There is an ad-hoc cost of taxation and an ad-hoc cost of repudiation that depends on the volume of debt. The focus of Calvo (1998) is on multiple equilibria, which result from his assumption on repudiation costs. With such a model, it is not possible to understand how debt, either nominal or indexed, can be used as a state variable in affecting future monetary policy; how optimal equilibrium paths should evolve, or why different welfare rankings of indexed versus nominal debts are possible.

Obstfeld (1997) and Ellison and Rankin (2005) assume that debt is real, and focus on monetary policy. They compute Markov-perfect equilibria when the source of the time inconsistency of monetary policy is related to the depletion of the real value of money balances. Obstfeld (1997) uses a model where money balances are not predetermined and therefore must consider an ad-hoc cost of a surprise inflation. Ellison and Rankin (2005) use the model in Nicolini (1998) with a class of preferences for which the level of real debt matters for the direction of the time inconsistency problem.

Martin (2006) analyzes a version of the same model we analyze, in which the government only issues nominal debt, not indexed. He provides an analytical characterization of the long-run behavior of Markov-perfect equilibria in the case of nominal debt, and shows that the long-run behavior depends on the intertemporal elasticity of substitution. Our paper analyzes and contrasts both types of debt regimes, providing a numerical comparison of Markov-perfect equilibrium outcomes, characterizing the equilibria and comparing the indexed and nominal debt regimes in terms of welfare.



A different strand of related literature is on how optimal policies under commitment can be made time consistent by properly managing the portfolio of government assets and liabilities. The closest paper to ours is Persson, Persson and Svensson (2006)[1]. They use a structure similar to Nicolini (1998) and assume that the government can use both nominal and real debt and that there are no restrictions on debt being positive or negative.

Although, we use as benchmark economies with full commitment, our main focus is on Markov-perfect equilibria. In fact, the full characterization and computation of the optimal policy in such equilibria – with debt as a state variable – is an additional contribution of our work.[2] Finally, there is a recent related literature on the characterization of the best sustainable equilibrium in similar optimal taxation problems, which also reaches the conclusion that optimal policies should, asymptotically, eliminate time inconsistency distortions (see, for example, Reis, 2006).

The paper proceeds as follows: In Section 2, we describe the model economy and define competitive equilibria with nominal and indexed debt. In Section 3, we characterize the optimal allocations and policies under commitment, for the purpose of understanding the sources of time inconsistency. In Section 4, we analyze and compute the Markov-perfect equilibria with indexed and nominal debt. Section 5 contains the main results of the paper: we compare the different regimes in terms of welfare. Finally in Section 6, we show that considering alternative taxes does not change the analysis, as long as those are set one period in advance.

## 2 The model economy

In this section we describe the model economy *with nominal debt*. We follow very closely the structure in Nicolini (1998). The economy is a production economy with linear technology,

$$c_t + g \leq n_t \tag{1}$$

for every $t \geq 0$, where $c_t$ and $g$ are private and public consumption, respectively and $n_t$ is labor. There is a representative household and a government. The preferences of the household are assumed to be linear in leisure and isoelastic in consumption,

$$\sum_{t=0}^{\infty} \beta^t [u(c_t) - \alpha n_t], \tag{2}$$

where $u(c) = \frac{(c)^{1-\sigma}-1}{1-\sigma}$. $0 < \beta < 1$ is the time discount factor.

[1] See also Alvarez, Kehoe and Neumeyer (2004) and Lucas and Stokey (1983).

[2] In this respect, our work is closely related to Krusell, Martín and Ríos-Rull (2003) who characterize the recursive equilibria that obtain in an optimal labor taxation problem, and other more recent work on Markov-perfect equilibria.

We assume that consumption in period $t$ must be purchased using currency carried over from period $t-1$ as in Svensson (1985). This timing of transactions implies that the representative household takes both $M_0$ and $B_0(1+i_0)$ as given and that an unexpected price increase is costly since it reduces planned consumption. The specific form of the cash-in-advance constraint faced by the representative household is:

$$P_t c_t \leq M_t \tag{3}$$

for every $t \geq 0$, where $P_t$ is the price of one unit of the date $t$ consumption good in units of money and $M_t$ are money balances acquired in period $t-1$ and used for consumption in period $t$.

Each period the representative household faces the following budget constraint:

$$M_{t+1} + B_{t+1} \leq M_t - P_t c_t + B_t(1+i_t) + P_t n_t \tag{4}$$

where $M_{t+1}$ and $B_{t+1}$ denote, respectively, the stock of money and the stock of nominal government debt that the household carries over from period $t$ to period $t+1$. The representative household faces a no-Ponzi games condition:

$$\lim_{T \longrightarrow \infty} \beta^T \frac{B_{T+1}}{P_T} \geq 0 \tag{5}$$

In each period $t \geq 0$, the government issues currency $M^g_{t+1}$ and nominal debt $B^g_{t+1}$, to finance an exogenous and constant level of public consumption $g$.[3] Initially, we abstract from any other source of public revenues. The sequence of government budget constraints is

$$M^g_{t+1} + B^g_{t+1} \geq M^g_t + B^g_t(1+i_t) + P_t g, \ t \geq 0 \tag{6}$$

where $i_t$ is the nominal interest rate paid on debt issued by the government at time $t-1$, together with the no-Ponzi games condition $\lim_{T \longrightarrow \infty} \beta^T \frac{B^g_{T+1}}{P_T} \leq 0$. The initial stock of currency, $M^g_0$, and initial debt liabilities, $B^g_0(1+i_0)$, are given. A government policy is, therefore, a specification of $\{M^g_{t+1}, B^g_{t+1}\}$ for $t \geq 0$.

### 2.1 A competitive equilibrium with nominal debt

**Definition 1** *A competitive equilibrium for an economy with nominal debt is a government policy, $\{M^g_{t+1},\ B^g_{t+1}\}^\infty_{t=0}$, an allocation $\{M_{t+1}, B_{t+1}, c_t, n_t\}^\infty_{t=0}$, and a price vector, $\{P_t, i_{t+1}\}^\infty_{t=0}$, such that:*

[3]We assume that government expenditures, $g$, are given, although our analysis can easily be extended to the case of endogenous government expenditures.



*(i) given $M^g_0$ and $B^g_0(1+i_0)$, and $g$, the government policy and the price vector satisfy the government budget constraints described in expression (6) together with a no-Ponzi games condition;*

*(ii) when households take $M_0$, $B_0(1+i_0)$ and the price vector as given, the allocation maximizes utility (2), subject to the cash-in-advance constraints (3), the household budget constraints (4), and the no-Ponzi games condition (5); and*

*(iii) all markets clear, that is: $M^g_{t+1} = M_{t+1}$, $B^g_{t+1} = B_{t+1}$, and $g$ and $\{c_t, n_t\}^\infty_{t=0}$ satisfy the economy's resource constraint (1), for every $t \geq 0$.*

Given our assumptions on the utility of consumption $u$, it is straightforward to show that the competitive equilibrium allocation of this economy satisfies both the economy's resource constraint (1) and the household's budget constraint (4) with equality, and that the first order conditions of the Lagrangian of the household's problem are both necessary and sufficient to characterize the solution to the household's problem. The cash-in-advance constraint, (3), is binding for $t \geq 0$ if $\frac{u_c(c_t)}{\alpha} > 1$. This condition is satisfied for $t \geq 1$ whenever $i_t > 0$, since $\frac{u_c(c_t)}{\alpha} = 1 + i_t$ for $t \geq 1$. For the first-best consumption, where $\frac{u_c(c_t)}{\alpha} = 1$, the cash-in-advance constraint does not have to hold with equality.

The competitive equilibrium of an economy with nominal debt can be characterized by the following conditions that must hold for every $t \geq 0$:

$$\frac{u_c(c_{t+1})}{\alpha} = 1 + i_{t+1}, \ t \geq 0, \tag{7}$$

$$1 + i_{t+1} = \beta^{-1} \frac{P_{t+1}}{P_t}, \ t \geq 0, \tag{8}$$

and the cash-in-advance constraint, which, if $\frac{u_c(c_t)}{\alpha} > 1, \ t \geq 0$, must hold with equality

$$c_t = \frac{M_t}{P_t}, \ t \geq 0, \tag{9}$$

Furthermore, the following equilibrium conditions must also be satisfied: the government budget constraints (6), the resource constraints (1) with equality, and the transversality condition (5)

$$\lim_{T \longrightarrow \infty} \beta^T \left( \frac{M_{T+1} + B_{T+1}}{P_T} \right) = 0 \tag{10}$$

implied by optimality given the no-Ponzi games condition (5).

### 2.2 An economy with indexed debt

An economy *with indexed debt* is an economy in all identical to the economy with nominal debt except in what concerns the government assets. The nominal interest rate adjusts with the price level so that $\frac{B_t(1+i_t)}{P_t} \equiv b_t$ is now predetermined for every period $t \geq 0$. The intertemporal budget constraint of the households can then be written as

$$M_{t+1} + \frac{b_{t+1}}{1+i_{t+1}} P_{t+1} \leq M_t - P_t c_t + b_t P_t + P_t n_t.$$

The first order conditions (7)-(9) are also first order conditions of the optimal problem with indexed debt.

A competitive equilibrium for an economy with indexed debt is defined as a government policy, $\{M^g_{t+1}, b^g_{t+1}\}_{t=0}^{\infty}$, an allocation $\{M_{t+1}, b_{t+1}, c_t, n_t\}_{t=0}^{\infty}$, and a price vector, $\{P_t, i_{t+1}\}_{t=0}^{\infty}$, such that the conditions (i), (ii) and (iii) of Definition 1 are satisfied when nominal liabilities are replaced by real liabilities, according to $\frac{B_t(1+i_t)}{P_t} = b_t$, where $b_t$ is predetermined.

### 2.3 Implementability with nominal debt

When choosing its policy the government takes into account the above equilibrium conditions. These conditions can be summarized with implementability conditions in terms of the allocations. In particular, as long as the cash-in-advance constraint is binding, the government budget constraint (6), which is satisfied with equality, can be written as the implementability condition

$$c_{t+1} u_c(c_{t+1}) \frac{\beta}{\alpha} + \beta z_{t+1} c_{t+1} = c_t + z_t c_t + g, \;\; t \geq 0 \tag{11}$$

where

$$z_t \equiv \frac{B^g_t(1+i_t)}{M^g_t}$$

To see this, notice that the budget constraint (6) with equality can be written in real terms as

$$\frac{M^g_{t+1}}{P_t} + \frac{B^g_{t+1}}{P_t} = \frac{M^g_t}{P_t} + \frac{B^g_t(1+i_t)}{P_t} + g \tag{12}$$

and, using the first order conditions of the households problem, (7), (8) and (9), as well as the cash-in-advance constraint with equality, $\frac{M^g_t}{P_t} = c_t$, one obtains the following identities: $\frac{M^g_{t+1}}{P_t} = \frac{M^g_{t+1}}{P_{t+1}} \frac{P_{t+1}}{P_t} = c_{t+1}\beta(1+i_{t+1}) = c_{t+1} u_c(c_{t+1}) \frac{\beta}{\alpha}$, $\frac{B^g_t(1+i_t)}{P_t} = \frac{B^g_t(1+i_t)}{M^g_t} \frac{M^g_t}{P_t} = z_t c_t$, and $\frac{B^g_{t+1}}{P_t} = \frac{B^g_{t+1}(1+i_{t+1})}{M^g_{t+1}} \frac{M^g_{t+1}/P_{t+1}}{P_t(1+i_{t+1})/P_{t+1}} = \beta z_{t+1} c_{t+1}$.



The intertemporal implementability condition (11) together with the terminal condition $\lim_{T\longrightarrow\infty} \beta^T (c_{T+1} u_c(c_{T+1}) \frac{\beta}{\alpha} + \beta z_{T+1} c_{T+1}) = 0$, obtained from the transversality condition (10), summarize the equilibrium conditions if the cash-in-advance constraint is always binding. Notice that the remaining equilibrium conditions are satisfied since equilibrium interest rates, prices, nominal liabilities and labor supplies can be derived from the competitive equilibrium restrictions; that is, $\{i_{t+1}\}_{t=0}^{\infty}$ satisfying (7), $\{P_{t+1}\}_{t=0}^{\infty}$ satisfying (8), $\{M_{t+1}\}_{t=0}^{\infty}$ and $P_0$ that satisfy (9), $\{n_t\}_{t=0}^{\infty}$ satisfying (1), $\{B_{t+1}\}_{t=0}^{\infty}$ so that $z_{t+1} = \frac{B_{t+1}(1+i_{t+1})}{M_{t+1}}$.

Using the terminal condition, the present value government budget constraint takes the form

$$\sum_{t=0}^{\infty} \beta^t \left( c_{t+1} u_c(c_{t+1}) \frac{\beta}{\alpha} - (c_t + g) \right) = z_0 c_0 \tag{13}$$

### 2.4 Implementability with indexed debt

With indexed debt, the government budget constraint (6 ) with equality can be written as the implementability condition

$$c_{t+1} u_c(c_{t+1}) \frac{\beta}{\alpha} + \beta b_{t+1} = c_t + b_t + g, \;\; t \geq 0, \tag{14}$$

provided the cash-in-advance constraint binds. The transversality condition (10) is written as $\lim_{T\longrightarrow\infty} \beta^T (c_{T+1} u_c(c_{T+1}) \frac{\beta}{\alpha} + \beta b_{T+1}) = 0$, which implies that the present value government budget constraint takes the form

$$\sum_{t=0}^{\infty} \beta^t \left( c_{t+1} u_c(c_{t+1}) \frac{\beta}{\alpha} - (c_t + g) \right) = b_0 \tag{15}$$

This condition summarizes, when debt is indexed, the competitive equilibrium restrictions on the sequence of consumption $\{c_t\}_{t=0}^{\infty}$. The only difference between the implementability conditions with nominal and indexed debt is in the right hand side of equations (13) and (15). With nominal debt, the government can affect the real value of outstanding debt, although this requires to affect consumption.

Notice also that an economy with nominal debt, and initial nominal liabilities $z_0$, where the government policy results in a choice of $c_0$ has the same period zero, *ex-post,* real liabilities than an indexed economy with initial –but, *predetermined*– real liabilities $b_0 = z_0 c_0$. We use such correspondence in comparing economies with nominal debt with economies with real debt.

## 3 Optimal policy with full commitment

In this section we compare optimal policies under full commitment when debt is indexed and when is nominal. This is useful because, by observing how the optimal allocations differ in the initial period from the subsequent ones, we are able to understand whether policy is time consistent, and when it is not, what is the source of the time inconsistency. In defining a full commitment Ramsey equilibrium with indexed debt, we assume –and, ex-post confirm– that the solution of the problem satisfies $\frac{u'(c_t)}{\alpha} > 1$, $t \geq 0$, so that the cash-in-advance constraint always binds[4].

**Definition 2** *A* ***full commitment Ramsey equilibrium with indexed debt*** *is a competitive equilibrium such that $\{c_t\}$ solves the following problem:*

$$Max \sum_{t=0}^{\infty} \beta^t \left[u(c_t) - \alpha(c_t + g)\right] \tag{16}$$

*subject to the implementability condition (15):*

$$\sum_{t=0}^{\infty} \beta^t \left( c_{t+1} u_c(c_{t+1}) \frac{\beta}{\alpha} - (c_t + g) \right) = b_0.$$

The other competitive equilibrium variables which are the government policy $\{M^g_{t+1}, B^g_{t+1}\}_{t=0}^{\infty}$, the allocation $\{M_{t+1}, B_{t+1}, n_t\}_{t=0}^{\infty}$, and the price vector, $\{P_t, i_{t+1}\}_{t=0}^{\infty}$, are obtained using the competitive equilibrium conditions.

The optimal solution with commitment results in a constant consumption path from period one on, $c_{t+1} = c_1$, $t \geq 0$. The intertemporal condition relating the optimal consumption in the initial period with the optimal stationary consumption in the subsequent periods is

$$u_c(c_0) - \alpha = \frac{u_c(c_1) - \alpha}{1 - \frac{u_c(c_1)}{\alpha}(1-\sigma)}. \tag{17}$$

Clearly, when $\sigma = 1$, the two consumptions are equal and the solution is time consistent. When $\sigma < 1$, (i.e., the intertemporal elasticity of substitution is $1/\sigma > 1$) the government prefers to tax more current consumption than future consumption, and, therefore, $c_0 < c_1$. When $\sigma > 1$, the government prefers to delay taxation and $c_0 > c_1$. In summary, when $\sigma \neq 1$, the full commitment solution is time-inconsistent due to 'intertemporal elasticity effects.'

In the definition of a full commitment Ramsey equilibrium, we have imposed a binding cash-in-advance constraint in all periods. When $\sigma \geq 1$ (i.e., $u_c(c_1) \geq u_c(c_0)$), the cash-in-advance constraint binds as long as there is a need to raise distortionary taxes. When $\sigma < 1$ (i.e., $u_c(c_1) < u_c(c_0)$), the cash-in-advance constraint binds as long as it is not possible to attain the first best from period $t = 1$ on[5].

**Definition 3** *A* ***full commitment Ramsey equilibrium with nominal debt*** *is a competitive equilibrium such that $\{c_t\}$ solves the following problem:*

$$Max \sum_{t=0}^{\infty} \beta^t \left[u(c_t) - \alpha(c_t + g)\right] \tag{18}$$

*subject to the implementability condition (13):*

$$\sum_{t=0}^{\infty} \beta^t \left( c_{t+1} u_c(c_{t+1}) \frac{\beta}{\alpha} - (c_t + g) \right) = z_0 c_0. \tag{19}$$

As in the economies with indexed debt, it is optimal for the government to commit to a constant path of consumption (and nominal interest rates) from period one on, but consumption in period zero may differ. With nominal debt, the intertemporal condition relating consumption in period zero and period one is given by

$$\frac{u_c(c_0) - \alpha}{1 + z_0} = \frac{u_c(c_1) - \alpha}{1 - \frac{u_c(c_1)}{\alpha}(1-\sigma)}. \tag{20}$$

This condition makes it explicit the additional motive for consumption in the two periods to diverge, when debt is nominal. Comparing (20) with the intertemporal condition with indexed debt (17), it can be seen that, as long as $z_0 \equiv \frac{B_0(1+i_0)}{M_0} > 0$, $c_0$ is relatively smaller –with respect to $c_1$– than the corresponding $c_0$ of the economy with indexed debt. In other words, the incentive to monetize debt always results in relatively lower period zero consumption; whether this results in lower consumption in the initial period with respect to the future consumption depends on how this effect interacts with the 'intertemporal elasticity' effect already present in the indexed economy. As in the economies with indexed debt, the government commits to a constant path of consumption (i.e., of nominal interest rates) from period one on, but consumption in period zero may be different, due to the 'intertemporal elasticity effect' (as in the indexed debt case) or to the 'nominal effect' of monetizing nominal debts and revaluating nominal assets.

A closer inspection of (20) also shows that, for every $\sigma$, there is a $\overline{z}_0$ resulting in a constant optimal consumption path from period zero on and, therefore, policy is time consistent. Such $\overline{z}_0$ is obtained by solving, for $c$, the following steady state implementability condition: $\overline{c} u_c(\overline{c}) \frac{\beta}{\alpha} + \beta \overline{z}_0 \overline{c} = (1 + \overline{z}_0)\, \overline{c} + g$. That is, substituting $\overline{z}_0 = -\frac{u_c(\overline{c})}{\alpha}(1-\sigma)$, the steady state equation reduces to $\frac{\overline{c} u_c(\overline{c})}{\alpha}\left[1 - (1-\beta)\sigma\right] = \overline{c} + g$. Therefore, as long as

[4]See Appendix 1 for a discussion of equilibria with first-best outcomes and with non binding cash-in-advance constraints.

[5]See Appendix 1.

$(1-\beta)\,\sigma < 1$, there is a solution for $\bar{c}$ and, correspondingly, for $\bar{z}_0$. Notice that $\bar{z}_0$ is negative, zero or positive, depending on whether $\sigma < 1$, $\sigma = 1$ or $\sigma > 1$[6]

In summary, in our economies, with full commitment, it is optimal to set the same inflation tax rate from period one on, resulting in a stationary consumption path from that period. With a unitary intertemporal elasticity of substitution (log utility), the optimal policy with indexed debt is time consistent (Nicolini, 1998). The time consistency is lost, for those same preferences, if debt is nominal, since there is an additional reason to inflate in the initial period, to reduce the real value of outstanding nominal liabilities. Under more general preferences, there is a time inconsistency even when debt is indexed. Nominal debt in that case exacerbates the time inconsistency problem when the elasticity of substitution is greater than one, since it reinforces the incentives to inflate, and alleviates it when the elasticity is lower than one. If public debt is negative the incentive for the government is to revaluate these assets, so that the 'nominal effect' works in the opposite direction[7].

## 4 Markov-perfect monetary equilibria

In this section we consider that the government cannot commit to a future path of policy actions and, therefore, chooses its monetary policy sequentially. We restrict the analysis to the case where such sequential choices do not depend on the whole history up to period $t$ but can depend on the pay-off relevant state variables –as in Markov-perfect equilibria– and, therefore, sequential optimal choices are recursively given by stationary optimal policies. In particular, in the case with nominal debt, government policy is recursively defined by $c_t = C(z_t)$ and the corresponding state transition $z_{t+1} = \boldsymbol{z}'(z_t)$. Similarly, with indexed debt, government policy is recursively defined by $c_t = C(b_t)$ and the corresponding state transition $b_{t+1} = \boldsymbol{b}'(b_t)$. Agents have rational expectations and, therefore, their consumption plans are consistent with the government policy choices and the corresponding state transitions. Our definitions of *Markov-perfect monetary equilibria* take these elements into account.

### 4.1 Indexed debt

**Definition 4** *A Markov-perfect monetary equilibrium*[8] *with indexed debt is a value function $V(b)$ and policy functions $C(b)$and $\boldsymbol{b}'(b)$ such that $c = C(b)$ and $b' = \boldsymbol{b}'(b)$ solve*

$$V(b) = \max_{c,b'} \{u(c) - \alpha(c+g) + \beta V(b')\}$$

[6]Ellison and Rankin (2005) show that this possibility of having time consistent optimal policies, for specific initial real liabilities, can also occur with indexed debt for some forms of non CRRA preferences.

[7]See Appendix 1.

[8]As in the economies with full commitment, we assume –and ex-post verify- that the cash-in-advance constraint is always binding.



*subject to the implementability constraint*

$$C(b')u_c(C(b'))\frac{\beta}{\alpha} + \beta b' = c + g + b, \quad t \geq 0.$$

In order to characterize the Markov-perfect equilibrium, notice the first order condition for $c$ is:

$$u_c(c) - \alpha = \lambda, \tag{21}$$

and for $b'$,

$$V_b(b') + \lambda\left(\frac{C_b(b')u_c(C(b'))}{\alpha}(1-\sigma) + 1\right) = 0.$$

while the envelope condition is given by

$$V_b(b) = -\lambda$$

These equations imply the following intertemporal condition along an equilibrium path,

$$\frac{u_c(c')}{\alpha} - 1 = \left(\frac{u_c(c)}{\alpha} - 1\right)\left[1 + \frac{u_c(C(b'))}{\alpha}C_b(b')\,(1-\sigma)\right] \tag{22}$$

It follows that in the log case ($\sigma = 1$), the optimal policy is stationary; given any sustainable level of initial debt, $b_0$, such level is maintained and, correspondingly, the consumption path is constant.

It is also the case that the first best, where $\frac{u_c(c)}{\alpha} = 1$, is a steady state, independently of the value of $\sigma$. Notice also that at the first best, where the stationary level of debt is negative and large enough in absolute value to cover expenditures, an increase in the level of assets does not affect consumption $C_b(b') = 0$, for $b' \leq b^*$, where $b^* < 0$ is the level of assets corresponding to the first best[9].

Comparing the equilibrium intertemporal condition of the economy without commitment (22) with the corresponding condition of the economy with full commitment, (17), we see that the 'intertemporal elasticity effect,' when $\sigma \neq 1$, is weighted by $-C_b(b')$, which is the marginal decrease of consumption in response to an increase in debt, as a function of the level of debt. In the numerical computations of the Markov-perfect equilibria, $C_b(b')$ is always negative. With $C_b(b')$ negative, the intertemporal condition (22) implies that when $\sigma < 1$ the consumption path is increasing, while when $\sigma > 1$ the consumption path is decreasing.

We now turn to the numerical results. In Appendix 2 we discuss the choice of parameters and describe the computational algorithm. Figure 1 shows the optimal debt

[9]Technically, there is a need to consider positive transfers, meaning that there is free disposal of extra revenues by the government (see Appendix 1).

and consumption policies, $\boldsymbol{b}'(b)$ or, rather, $\boldsymbol{b}'(b)-b$, and $C(b)$, for $\sigma=0.6$, $\sigma=1.0$, and $\sigma=1.4$. As we have already mentioned, when $\sigma=1.0$ the initial level of debt is maintained forever; i.e., $\boldsymbol{b}'(b)-b=0$.

When $\sigma<1$ the policy function $\boldsymbol{b}'(b)$ is decreasing and $\boldsymbol{b}'(b)<b$, except at the value of debt that supports the first best level of consumption where there is no distortionary taxation[10] . The inflation tax is higher initially so that debt may be depleted and assets accumulated, to the point where there are enough assets to finance all expenditures, and the first best is attained. At the first best, the 'intertemporal elasticity effect' disappears.

When $\sigma>1$, the policy function $\boldsymbol{b}'(b)$ is increasing and $\boldsymbol{b}'(b)>b$, except at the first best level of debt $b^*<0$. Furthermore, $\frac{\boldsymbol{b}'(b)}{b}<\beta^{-1}$, so that debt is accumulated at a rate lower than $\beta^{-1}-1$. The first best is also a steady state when $\sigma>1$, but it is not the asymptotic state and, therefore, the 'intertemporal elasticity effect' never disappears. Panel D also shows that, when $\sigma=1.4$, $C_b(b')$ is very close to zero. In general, as our computations have also shown, whenever $\sigma$ is close to one, $C_b(b')$ is very close to zero and, correspondingly, $\boldsymbol{b}'(b)\approx b$. This means that the convergence to the first best when $\sigma<1$, or the accumulation of debt when $\sigma>1$, is very slow.

### 4.2 Nominal debt

As we have seen, in an economy with full commitment and nominal debt, there is a 'nominal effect' since the government has an incentive to partially monetize the debt in the initial period. In an economy without commitment, such distorting effect is present every period and, therefore, is anticipated by households. As in the economy with full commitment, such 'nominal effect' interacts with the possible 'intertemporal elasticity effect'.

**Definition 5** *A Markov-perfect monetary equilibrium with nominal debt is a value function $V(z)$ and policy functions $C(z)$ and $\boldsymbol{z}'(z)$ such that $c=C(z)$ and $z'=\boldsymbol{z}'(z)$ solve*

$$V(z)=\max_{\{c,z'\}}\{u(c)-\alpha(c+g)+\beta V(z')\} \tag{23}$$

*subject to the implementability constraint*

$$C(z')u_c(C(z'))\frac{\beta}{\alpha}+\beta z'C(z')=zc+c+g. \tag{24}$$

[10] Notice that policies show small high frequency fluctuations due to our discrete grid algorithm. However, such fluctuations do not impinge on our results and, to gain clarity, we also show in Figure 1 (Panels A, B, E and F) fitted a four degree polynomials; see Appendix 2, which also provides the values of debt supporting the first best.



To characterize the Markov-perfect monetary equilibrium, notice that the first order conditions of the problem described above are

$$u_c(c)-\alpha=\lambda(1+z) \tag{25}$$

and

$$V_z(z')+\lambda\left(\frac{C_z(z')u_c(C(z'))}{\alpha}(1-\sigma)+C(z')\left[1+\varepsilon_c(z')\right]\right)=0 \tag{26}$$

where $\varepsilon_c(z)=\frac{zC_z(z)}{C(z)}$ is the elasticity of $C(z)$, and the envelope condition is

$$V_z(z)=-\lambda c \tag{27}$$

This implies

$$\frac{\frac{u_c(c')}{\alpha}-1}{1+z'}=\frac{\frac{u_c(c)}{\alpha}-1}{1+z}\left(1+\varepsilon_c(z')+\frac{u_c(C(z'))C_z(z')}{\alpha C(z')}(1-\sigma)\right) \tag{28}$$

For $z'\neq 0$, this can be written as

$$\frac{\frac{u_c(c')}{\alpha}-1}{1+z'}=\frac{\frac{u_c(c)}{\alpha}-1}{1+z}\left(1+\varepsilon_c(z')\left[1+\frac{u_c(C(z'))}{z'\alpha}(1-\sigma)\right]\right) \tag{29}$$

As in the case with indexed debt, there is a first best steady state, with $\frac{u_c(c)}{\alpha}=1$, where government assets are enough to finance expenditures. If $z\equiv z^*=-1-\frac{g}{(1-\beta)c^*}$ with $u_c(c^*)=\alpha$, the cash-in-advance constraint holds with equality and the solution is the first best. This is an isolated steady state[11]**.**

As we have seen in the previous section, with nominal debt there is another steady state, where $\overline{z}=-\frac{u_c(\overline{c})}{\alpha}(1-\sigma)$, and $\overline{c}$ solves $\frac{\overline{c}u_c(\overline{c})}{\alpha}[1-(1-\beta)\sigma]=\overline{c}+g$.

To better understand the distortions present in the economy with nominal debt without commitment, it is useful to consider the log case, where the intertemporal condition (28) can be rewritten as

$$\frac{\frac{1}{c}-\alpha}{\left[1+\frac{(1+i)B}{M}\right]}=\frac{\frac{1}{c'}-\alpha}{\left[1+\frac{(1+i')B'}{M'}\right]}\left[1+\varepsilon_c\left(\frac{(1+i')B'}{M'}\right)\right]^{-1} \tag{30}$$

where the elasticity $\varepsilon_c(z')$ is negative and less than one in absolute value (as our computations show).

[11] See the Appendix 1 for a more detailed discussion of the case $z\leq -1$.

This intertemporal equation reflects the different distortions present as a result of debt being nominal and policy decisions being sequential. The term $\left[1+\frac{(1+i)B}{M}\right]$ results from the discretionary incentive to reduce the real value of nominal debt. It is present in the problem with commitment only in the initial period (equation (20)). The marginal benefit of increasing current consumption is discounted by the current liabilities, $\frac{(1+i)B}{M}$, reflecting the fact that a higher consumption in the current period implies a higher real value of outstanding nominal debt and, therefore, higher future distortionary taxes. Hence, the benefits for the benevolent government of higher consumption today must be discounted to take into account these future costs. The term $\left[1+\varepsilon_c\left(\frac{(1+i')B'}{M'}\right)\right]$ results from the dynamic nature of this problem and reflects the cost of exacerbating the time inconsistency problem in the future, due to an increase in outstanding liabilities at the end of the current period[12]. For $\sigma \neq 1$, these two effects are compounded with the 'intertemporal elasticity effects' and, as we have seen, the interaction of all these effects may result in stationary solutions not present in the economy with indexed debt.

In Figure 2 we report our findings for the same three elasticity values, $\sigma = 0.6$, $\sigma = 1.0$, and $\sigma = 1.4$, when debt is nominal. We find that in all three cases, $\boldsymbol{z}'(z) - z$ is decreasing and, as we have shown before, there is a steady-state at $\overline{z} = 0$ when $\sigma = 1$, $\overline{z} < 0$ when $\sigma < 1$, and $\overline{z} > 0$ when $\sigma > 1$[13]. Since $\boldsymbol{z}'(z) - z$ is decreasing, these steady states correspond to the asymptotic behavior of nominal debt paths.

In contrast with the indexed debt case illustrated in Figure 1, when $\sigma = 1$ the stock of debt is no longer constant, but it is progressively depleted until the 'nominal effect' disappears at $\overline{z} = 0$. When $\sigma < 1$, debt is also progressively depleted and assets are accumulated but, in contrast with the indexed debt case, this process leads to a level of assets $\overline{z}$ which is lower than the first best steady state level since $-1 < \overline{z} < 0$ and the first best would require $z < -1$.

When $\sigma > 1$, debt is not accumulated without bound —as was the case with indexed debt— but it is accumulated or progressively depleted until it reaches the distorted steady state $\overline{z} > 0$ in which the 'nominal' and the 'intertemporal elasticity' effects cancel out. As in the indexed debt case, for $\sigma = 1.4$, $\boldsymbol{z}'(z) - z$ is fairly flat and, as a result, long-run convergence (divergence in the indexed debt case) is very slow (see Panel E in Figure 1).

## 5 Welfare comparisons

In the last two sections we have seen how optimal monetary policies may result in different time paths for debt and inflation, corresponding to different distortions, depending on whether monetary authorities can, or cannot, commit and whether debt is indexed or

[12]Myopic governments that would not take into account the effect of their choices on the state variables of future government decisions would be solving a problem where only the first term would be present.

[13]See Appendix 2 for the corrpesponding values of $\overline{z}$.



nominal. In this section we address the central question of how these different monetary regimes compare in terms of welfare.

There is an immediate and unambiguous comparison between economies with and without commitment with the same type of liabilities (i.e., either indexed or nominal debt in both economies). As one should expect, Markov-perfect equilibria are less efficient than full commitment Ramsey equilibria, unless the Ramsey equilibria are stationary and, therefore, time consistent. This result follows from the fact that the full commitment Ramsey solution is the choice of a sequence of consumption $\{c_t\}_{t=0}^{\infty}$ which maximizes utility (16) in the set of competitive equilibrium sequences defined either by (15) with indexed debt, or (13) with nominal debt, while the Markov-perfect equilibrium imposes additional restrictions to this maximization problem: the optimality of decisions of future governments. In other words, the Ramsey solution is the solution of a maximization problem of a committed government in period zero, while the Markov-perfect equilibrium can be better thought as the equilibrium of a game between successive governments, in which the optimal plan of the period zero government has to take into account the sequential decisions of future governments (or of his own future revised policies).

It is less straightforward to compare economies with different types of liabilities. Nevertheless, we can compare the welfare of an economy with nominal debt with an –otherwise identical– economy with indexed debt, provided that both have the same initial (equilibrium) levels of either real or nominal liabilities.

We compare economies with the same real value of initial liabilities. In economies with full commitment, if in the indexed debt economy initial real liabilities are $b_0$, then in the nominal debt economy nominal liabilities $z_0$ have to be such that $b_0 = z_0 C_0(z_0)$, where $C_0(z_0)$ is the full commitment optimal choice of initial consumption in the economy with nominal debt[14]. It is then trivial to compare the two economies in terms of welfare under full commitment. Given that both economies have the same initial real liabilities $b_0$, a benevolent government accounting for such real liabilities achieves higher welfare than a planner who does not, i.e., welfare in the indexed debt economy is higher than in the corresponding nominal debt economy. We state this formally in the following proposition.

**Proposition 1** *Consider two economies with full commitment with an initial money stock $M_0$. One of them has initial nominal debt $B_0(1+i_0) > 0$, and the other has initial indexed debt $b_0$. Suppose $b_0 = \frac{B_0(1+i_0)}{P_0}$, where $P_0$ is the period zero price in the economy with nominal debt. Then, the economy with nominal debt always gives lower welfare independently of the value of $\sigma$.*

**Proof:** The nominal debt economy has initial condition $z_0 = \frac{B_0(1+i_0)}{M_0}$, while the indexed economy has initial condition $b_0 = z_0 C_0(z_0)$, where $C_0(z_0)$ is the full commitment

[14]If the mapping from $b_0$ to $z_0$ is not unique, we select the lowest $z_0$; however, it is unique in our computations.

optimal choice of initial consumption in the economy with nominal debt. The optimal solutions in the two economies has to satisfy the same implementability condition, taking into account that optimal consumption paths are constant after period zero,

$$\beta c_1 \left[ \frac{u_c(c_1)}{\alpha} - 1 \right] - (1-\beta)\, c_0 - g = (1-\beta)\, b_0.$$

Given this condition the solution with the highest welfare is the solution with indexed debt, satisfying

$$u_c(c_0) - \alpha = \frac{u_c(c_1) - \alpha}{\left(1 - \frac{u_c(c_1)}{\alpha}(1-\sigma)\right)}.$$

since the solution with nominal debt is distorted by $z_0$ (even if there is no 'unexpected inflation'; i.e., realized real liabilities are $b_0$) according to

$$\frac{u_c(c_0) - \alpha}{1 + z_0} = \frac{u_c(c_1) - \alpha}{\left(1 - \frac{u_c(c_1)}{\alpha}(1-\sigma)\right)}.$$

If the solution with nominal debt is different, as is the case as long as $z_0 \neq 0$, then the solution with nominal debt gives strictly lower welfare.■

The choice of comparing economies with the same real value of initial liabilities under full commitment is consistent with how the Markov-perfect equilibria of the two economies without full commitment should be compared. The Markov-perfect equilibrium, as any competitive equilibrium, imposes the 'rational expectations' condition that agents' have the right expectations regarding future liabilities. In particular, the indexed economy is characterized by having nominal interest rates adjusting to price changes so as to guarantee the predetermined value of real liabilities. With nominal debt, real liabilities are not predetermined, but with rational expectations (and no uncertainty), *ex-post* real returns correspond to agents' *ex-ante* expected values. In a Markov-perfect equilibrium, strategies are policies that only depend on the state variable and, therefore, such policies do not treat differently period zero. It follows that, in a Markov-perfect equilibrium of an economy with nominal debt the government has no 'free lunch' from unexpected inflation, even in period zero.

As we have just seen, with full commitment indexed debt is unambiguously more efficient than nominal debt, when the pure monetization effect is not considered. Without commitment, along a Markov-perfect equilibrium path, there are no 'free lunches'; does this mean that indexed debt is better than nominal debt? The following proposition provides an interesting answer to this question.

**Proposition 2** *Consider two economies without commitment and initial money stock* $M_0$. *One of them has initial nominal debt* $B_0(1+i_0) > 0$, *and the other has initial indexed debt* $b_0$. *Suppose* $b_0 = \frac{B_0(1+i_0)}{P_0}$, *where* $P_0$ *is the period zero price in the economy with nominal debt.*



(*i*) *If* $\sigma = 1$, *the welfare in the economy with indexed debt is higher than in the economy with nominal debt.*

(*ii*) *If* $\sigma \neq 1$, *the welfare in the economy with nominal debt can be higher, or lower, than in the economy with indexed debt, depending on* $b_0$.

Part (*i*) of Proposition 2 follows from previous results. First, in the log case with indexed debt, policy is time consistent; hence, the full commitment and the Markov-perfect equilibria coincide. Second, from Proposition 1, the full commitment equilibrium with nominal debt provides lower welfare than the equilibrium with indexed debt. Third, as we have seen, the Markov-perfect equilibrium introduces additional constraints to the full-commitment maximization problem, resulting in lower welfare in the economy with nominal debt. Therefore, when $\sigma = 1$, indexed debt dominates nominal debt in terms of welfare. Part (*ii*) of Proposition 2 follows from our numerical simulations, showing that welfare reversals may occur.

Figure 3 shows that –in spite of the roughness of our discrete choice algorithm– the value functions are very well behaved; e.g., decreasing and concave in $b$ and fairly smooth. This allows for (robust) welfare comparisons, once indexed and nominal debt value functions take the same domain. Figure 4 illustrates Proposition 2. It compares the value functions with indexed debt $V_i$ and with nominal debt $V_n$ as functions of nominal liabilities $z$; that is, $V_i(\boldsymbol{b}(z))$, where $\boldsymbol{b}(z) \equiv zC(z)$.

Panels C and D of Figure 4 show the unambiguous result in part (*i*) of Proposition 2 that, with unitary elasticity, indexed debt dominates nominal debt in terms of welfare. The remaining panels illustrate the result in the second part of Proposition 2. In particular, when $\sigma \neq 1$, there is a range of assets and debts for which nominal debt Pareto dominates indexed debt.

There are two effects at play. First, because we compare economies with the same initial real liabilities, indexed debt tends to give higher welfare than nominal debt. This is the case under full commitment as stated in Proposition 1, but the effect is also present in Markov-perfect equilibria. The second effect that could either compensate or reinforce the first one, is the magnitude of the dynamic distortions induced by the time inconsistency.

With $\sigma = 1$, Ramsey policy with indexed debt is time consistent, while it is time inconsistent with nominal debt. In a Markov-perfect equilibrium, nominal debt is depleated to the point where debt is zero and policy is time consistent. Such departure from stationarity is costly and, as a result, indexed debt dominates nominal debt, in terms of welfare, when $\sigma = 1$.

If $\sigma \neq 1$, Ramsey policy is time inconsistent when debt is indexed while when debt is nominal there is a level of nominal debt (or assets) such that policy is time consistent. In particular, as we have seen, when $\sigma < 1$ and $z < 0$ (or, alternatively, when $\sigma > 1$ and $z > 0$) the 'intertemporal elasticity effect' and the 'nominal effect' tend to mutually offset; in fact, both effects fully cancel out at the distorted steady state $\overline{z} < 0$

(alternatively, $\overline{z} > 0$). At the distorted steady state, $\overline{z}$, the elimination of the time inconsistency distortions more than compensates the potential dominance of indexed debt. As a result, welfare is higher with nominal debt. Such a dominance of nominal debt is still present in a range of initial debt (or asset) levels close to the distorted steady states. This follows from the fact that Markov-equilibrium paths converge to –the nearby– distorted steady state and, therefore, the cost of anticipated distortions –due to 'time inconsistencies'– is relatively small. In fact, the dominance of nominal debt can still be present at initial values of debt for which the 'intertemporal elasticity effect' and the 'nominal effect' actually reinforce each other; that is, for relatively low debt values, $z > 0$, when $\sigma < 1$, or for relatively low asset values, $z < 0$, when $\sigma > 1$, as can be seen Figure 4.

As the intertemporal equilibrium condition (30) shows, the time-inconsistency 'nominal effect' is exacerbated by the size of $z$ (in absolute value); i.e. by the debt (or asset) to money ratio. Therefore, for large values of $z$ (in absolute value) the dynamic distortions due to the time inconsistency are very costly. It follows that, for high levels of debt (or assets), relative to money, indexed debt dominates nominal debt in terms of welfare.

When $\sigma < 1$, debt depletion is a characteristic of Markov-perfect equilibria with both indexed and nominal debt. If the initial value of debt is high, the relative advantage of converging to $\overline{z} < 0$, as opposed to the first best, is properly discounted into the distant future and offset by the more immediate cost of having the 'nominal effect' reinforcing the 'intertemporal elasticity effect'. Similarly, when $\sigma > 1$ and the initial level of debt is much higher than $\overline{z} > 0$, the relative advantage of converging to $\overline{z} > 0$ is properly discounted and offset by the cost of depleting the debt, given that the 'intertemporal elasticity effect' calls for postponing taxation and accumulating debt; a recommendation that sequential optimal policy follows with indexed debt, but not with nominal debt. It should also be noticed that a similar effect can happen when $\sigma < 1$ and $-1 < z < \overline{z} < 0$, then the 'intertemporal elasticity effect' calls for anticipating taxation and accumulating assets, while the 'nominal effect' calls for a relatively costly process of asset depletion; as a result, in this case, welfare with indexed debt is higher when assets are high (see Panel B.)

As summary we emphasize two points. The first is that we provide an interesting example of the principle that adding a distortion to a second best problem can actually improve welfare. Nominal debt adds a dynamic distortion to the Markov-perfect equilibrium. In the case where policy is time consistent with indexed debt, adding this distortion reduces welfare. When policy is time inconsistent with indexed debt, there are two dynamic distortions, due to the differing elasticities and to nominal debt. In this case, adding the distortion from nominal debt can actually raise welfare.

The second has important policy implications. In the calibrated economy with $\sigma$ different from one, when debt is relatively high (relative to output) it is better to have indexed debt, but for moderate levels of debt, it is preferable to have nominal debt; i.e., it is better to converge to the level of debt associated with the - nominal debt - distorted



steady state (positive or negative), rather than have an indefinite accumulation of debt or its depletion and subsequent accumulation of assets all the way to the first best.

## 6 Additional taxes

In most advanced economies, seigniorage is a minor source of revenue, and government liabilities are financed mostly through consumption and income taxes. This raises the question of whether our previous analysis is still valid when taxes are introduced, so as to make seigniorage a marginal component of governments' revenues. There is an additional motivation to introduce taxes in our model, that is to inquire whether a fully committed fiscal authority can overrun the commitment problems of a monetary authority. In this section we address these questions by introducing both consumption and labor income taxes.

We show first that the introduction of taxes, while reducing the need to raise revenues through seigniorage, does not change the characterization of equilibria with respect to the economies without taxes, analyzed in the previous sections. This is the case if the fiscal authority simply sets taxes one period in advance and, subsequently, the monetary authority sets its policy. This is a reasonable assumption taking into account the different frequencies in which monetary and fiscal decisions are typically made and implemented. Second, we show that, if the intertemporal elasticity of substitution is not lower than one and there is full commitment on the part of the fiscal authority (who makes policy choices before the monetary authority does) it is part of an optimal fiscal policy to finance all the outstanding government liabilities with taxes, constraining the monetary authority to implement the Friedman rule of a zero nominal interest rate, and implement the full commitment Ramsey solution with indexed debt, independently on whether debt is indexed or nominal and on whether there is full commitment or not on the monetary authority's side.

When the government levies consumption and labor income taxes, the household problem becomes:

$$\max \sum_{t=0}^{\infty} \beta^t \left[ u(c_t) - \alpha n_t \right] \tag{31}$$

subject to:

$$P_t(1+\tau_t^c)c_t \leq M_t \tag{32}$$

$$M_{t+1} + B_{t+1} \leq M_t - P_t(1+\tau_t^c)c_t + B_t(1+i_t) + P_t(1-\tau_t^n)n_t, \tag{33}$$

where $\tau_t^c, \tau_t^n$ are consumption and labor income taxes, respectively, and subject to the terminal condition:

$$\lim_{T\longrightarrow\infty}\beta^{T}\frac{B_{T+1}}{P_{T}}\geq 0 \tag{34}$$

Now, the marginal conditions (7), (8) and (9) characterizing the households's optimal choice become:

$$\frac{u_c(c_{t+1})}{\alpha}=(1+i_{t+1})\frac{1+\tau^c_{t+1}}{1-\tau^n_t} \tag{35}$$

$$1+i_{t+1}=\beta^{-1}\frac{P_{t+1}\left(1-\tau^n_{t+1}\right)}{P_t\left(1-\tau^n_t\right)} \tag{36}$$

and

$$c_t=\frac{M_t}{P_t(1+\tau^c_t)} \tag{37}$$

These conditions must hold for every $t\geq 0$. Notice that (35) reflects the fact that the household makes plans based on expectations about both interest rates and taxes. The intertemporal condition (36) introduces labor income taxes into expression (8), while the cash-in-advance constraint (37) now includes consumption taxes.

The sequence of government budget constraints in this economy is now given by:

$$P_tg+M^g_t+B^g_t(1+i_t)\leq P_t\tau^c_tc_t+P_t\tau^n_tn_t+M^g_{t+1}+B^g_{t+1} \tag{38}$$

while the feasibility conditions (1) do not change.

Define the *effective labor income tax* as:

$$\tau_t=\frac{\tau^c_t+\tau^n_t}{1+\tau^c_t},\text{ i.e., }(1-\tau_t)=\frac{1-\tau^n_t}{1+\tau^c_t}$$

Then, dividing the government budget constraint by $P_t(1-\tau^n_t)$, and noticing that $\frac{M^g_t}{P_t(1-\tau^n_t)}=c_t(1-\tau_t)^{-1}$, one obtains the following identities: $\frac{M^g_{t+1}}{P_t(1-\tau^n_t)}=\frac{M^g_{t+1}}{P_{t+1}(1-\tau^n_{t+1})}\frac{P_{t+1}(1-\tau^n_{t+1})}{P_t(1-\tau^n_t)}=c_{t+1}(1-\tau_{t+1})^{-1}\beta(1+i_{t+1})=c_{t+1}u_c(c_{t+1})\frac{\beta}{\alpha}$; $\frac{B^g_t(1+i_t)}{P_t(1-\tau^n_t)}=\frac{B^g_t(1+i_t)}{M^g_t}\frac{M^g_t}{P_t(1-\tau^n_t)}=z_tc_t(1-\tau_t)^{-1}$, and $\frac{B^g_{t+1}}{P_t(1-\tau^n_t)}=\frac{B^g_{t+1}(1+i_{t+1})}{M^g_{t+1}}\frac{M^g_{t+1}/P_{t+1}(1-\tau^n_{t+1})}{P_t(1-\tau^n_t)(1+i_{t+1})/P_{t+1}(1-\tau^n_{t+1})}=\beta z_{t+1}c_{t+1}(1-\tau_{t+1})^{-1}$. Therefore, the implementability conditions (11) can be written as

$$c_{t+1}u_c(c_{t+1})\frac{\beta}{\alpha}+\beta z_{t+1}c_{t+1}(1-\tau_{t+1})^{-1}=c_t+z_tc_t(1-\tau_t)^{-1}+g \tag{39}$$

together with the terminal condition

$$\lim_{T\longrightarrow\infty}\beta^{T}c_{T+1}u_c(c_{T+1})\frac{\beta}{\alpha}+\beta z_{T+1}c_{T+1}(1-\tau_{T+1})^{-1})=0 \tag{40}$$



### 6.1 Optimal monetary policy when the fiscal authority moves one period in advance

We now consider the case where tax decisions for some period $t$ must be made one period in advance, and may depend only on the state at $t-1$. In this case we can define the new state variable $\widehat{z}_t\equiv z_t(1-\tau_t)^{-1}$, and the problems are isomorphic to the problems in the previous sections, since the implementability condition (39) reduces to

$$c_{t+1}u_c(c_{t+1})\frac{\beta}{\alpha}+\beta\widehat{z}_{t+1}c_{t+1}=c_t+\widehat{z}_tc_t+g \tag{41}$$

which is formally identical to (11).

There is an additional restriction that the nominal interest rate must be nonnegative. This constraint was satisfied when seigniorage was the only source of revenue, but it is not necessarily satisfied in this case.

Notice that financing liabilities through taxes reduces the need to use seigniorage, however this does not necessarily mean distortions are reduced since, for example $u_c(c_{t+1})/\alpha=(1+i_{t+1})(1-\tau_{t+1})^{-1}$. Similarly even if, for a given $z_0>0$, the corresponding sequence $\{z_{t+1}\}^{\infty}_{t=0}$ can be lower than in an economy without taxes, the relevant state is $\widehat{z}_t\equiv z_t(1-\tau_t)^{-1}$, which is magnified by the presence of taxes.

Consistent with our framework, we assume that the fiscal authority either sets a sequence of taxes $\{\tau_t\}^{\infty}_{t=0}$ – and commits to it – or defines a recursive strategy for taxes – from period one on – as a function of the state; i.e., sets $\tau_0$ and $\tau_{t+1}=\tau(z_t)$. As long as the nominal interest rates are away from the lower bound of zero nominal interest rates, the problem for the monetary authority has the same structure as before and, therefore, the same results go through, even if part of the government liabilities are financed with taxes. To see this, notice that $u_c(c_{t+1})/\alpha=(1+i_{t+1})(1-\tau_{t+1})^{-1}$ and, as long as $(1-\tau_{t+1})u_c(C(z_{t+1}\left(1-\tau_{t+1})^{-1}\right))/\alpha>1$, the recursive strategies of the monetary authority - $C(\widehat{z})$ and $\boldsymbol{z}'(\widehat{z})$ - are the same as in the economy without taxes.

In summary, the monetary authority faces the same problem with consumption and income taxes than with only seigniorage, for any degree of monetary commitment. Therefore, the allocations for consumption and labor, for the various types of debt and monetary policy commitment technologies, are exactly the same as those that were obtained before.

**Additional taxes with indexed debt.** In the economy with indexed debt let

$$\widehat{b}_t=b_t(1-\tau^n_t)^{-1}=\frac{B_t(1+i_t)}{P_t(1-\tau^n_t)}$$

That is, $\widehat{b}_t = z_t c_t (1-\tau_t)^{-1} = \widehat{z}_t c_t$, then the implementability condition with indexed debt can be written as

$$c_{t+1} u_c(c_{t+1}) \frac{\beta}{\alpha} + \beta \widehat{b}_{t+1} = c_t + \widehat{b}_t c_t + g. \tag{42}$$

If $\tau_t^n = 0$, than this implementability condition is equivalent to (14) with $\widehat{b}_t = b_t$. This means that when there are only consumption taxes, whether these are decided one period in advance or not, consumption taxes do not change the policy problem and therefore the optimal path of indexed debt. With labor taxes set one period in advance, as long as nominal interest rates are positive, the policy choices $C(\widehat{b})$ and $\boldsymbol{b}'(\widehat{b})$ are the same.

### 6.2 When the fiscal authority can fully commit

Whenever the discretionary incentive is to have a higher current tax (always the case with $\sigma \leq 1$), it is possible to set taxes in a way that the resulting monetary policy follows the Friedman rule of zero nominal interest rates[15]. If there is commitment to fiscal policy, even without commitment to monetary policy, it is possible to implement the full commitment solution of the corresponding indexed debt economy.

To see this, consider an economy with nominal debt and initial liabilities $\widehat{z}_0$. As we have seen in Section 3, if in such economy there are no taxes and the monetary authority can fully commit, the equilibrium allocation is given by $(\boldsymbol{c}_0(\widehat{z}_0), c_t = \boldsymbol{c}_1(\widehat{z}_0), t \geq 1)$, and the corresponding period zero real liabilities are $\widehat{b}_0 = \boldsymbol{b}_0(\widehat{z}_0) \equiv \widehat{z}_0 \boldsymbol{c}_0(\widehat{z}_0)$.

We then take as reference the indexed economy without taxes and initial real liabilities $\widehat{b}_0$. The corresponding full commitment equilibrium is characterized by a consumption allocation $(\boldsymbol{c}_0(\widehat{b}_0), \boldsymbol{c}_1(\widehat{b}_0))$ and a constant interest rate path –from period one on – given by $(1 + i(\widehat{b}_0)) = u_c(\boldsymbol{c}_1(\widehat{b}_0))/\alpha$. We use such economy to define the fiscal policy in our original nominal economy with taxes and a monetary authority which may be unable to commit. In particular, let the fiscal authority set for $t \geq 0$, $\tau_{t+1} = \tau(\widehat{z}_0)$, where $(1 + \tau(\widehat{z}_0)) = (1 + i(\boldsymbol{b}_0(\widehat{z}_0)))^{-1}$. If, at any $t > 0$, the monetary authority tries to monetize part of the existing stock of nominal debt and to use the resulting revenues to increase future consumption —for example, maintaining a constant $\bar{c}$ from then on— then, it must be the case that $c_t < c_1(\boldsymbol{b}_0(\widehat{z}_0)) < \bar{c}$. However, in order to have

$$\frac{1+\bar{i}}{1+\tau(\widehat{b}_0)} = \frac{u_c(\bar{c})}{\alpha}$$

and $(1 + \tau(\widehat{z}_0))^{-1} = u_c(\boldsymbol{c}_1(\boldsymbol{b}_0(\widehat{z}_0)))/\alpha$, the interest rate would have to be negative, $\bar{i} < 0$.

[15] Notice, the Friedman rule would restore efficiency in an economy with cash and credit goods, since it would eliminate the distortion between these two types of goods created by the cash-in-advance constraint.



Negative interest rates can not be an equilibrium in this economy since then the household would like to borrow unboundly. Once it is not possible to raise future consumption with negative taxes, there is no gain in partially monetizing the stock of nominal debt in period zero. In this case monetary policy passively helps to implement the full commitment solution of the corresponding indexed debt economy. Therefore if there is no commitment to monetary policy, a fully committed fiscal authority who wants to maximize utility (2), sets $\tau_{t+1} = \tau(\widehat{z}_0)$, $t \geq 0$.[16] .

Notice that the above argument is simpler if one considers, at the outset, an indexed debt economy with initial liabilities $\widehat{b}_0$; therefore, it also applies to the case with indexed debt, $\sigma < 1$, and a monetary authority who sets its policy sequentially.

**Proposition 3** *In an economy with either nominal or indexed debt (with $\widehat{z}_0$ or $\widehat{b}_0$) assume fiscal authorities maximize the welfare of the representative household and fully commit to their policies. If $\sigma \leq 1$ then the fiscal authorities can induce the implementation of the full commitment equilibrium allocation of the corresponding indexed debt economy (i.e.,$\boldsymbol{b}_0(\widehat{z}_0)$ or $\widehat{b}_0$) regardless of the degree of commitment of the monetary authority.*

As we have seen, when $\sigma > 1$, the monetary authority can have an incentive to reduce the current price level (i.e., the 'intertemporal elasticity' effect may dominate the 'nominal debt' effect) and, in such case, the previous argument can not be applied.

## 7 Concluding remarks

This paper discusses the different ways in which nominal and indexed debt affect the sequential choice of optimal monetary and debt policies. To this purpose, we study a general equilibrium monetary model where the costs of an unanticipated inflation arise from a cash-in-advance constraint with the timing as in Svensson (1985), and where government expenditures are exogenous.

In our environment, as in Nicolini (1998), when the utility function is logarithmic in consumption and linear in leisure and debt is indexed, there is no time-inconsistency problem. In this case, the optimal monetary policy is to maintain the initial level of indexed debt, independently of the level of commitment of a benevolent government.

In contrast, for the same specification of preferences, when the initial stock of government debt is nominally denominated, a time inconsistency problem arises. In this case, the government is tempted to inflate away its nominal debt liabilities. When the government cannot commit to its planned policies, progressively depleting the outstanding stock of debt is part of an optimal sequential policy consists, so that such policy converges asymptotically to zero debt liabilities. Optimal nominal interest rates in this case are also decreasing and converge asymptotically.

[16] Marimon, Nicolini, and Teles (2003) make a similar argument in a different context.

In the rational expectations equilibria of our economies there are no surprise inflations. Still, for a given initial real value of outstanding debt, the sequential optimal equilibrium with indexed debt provides higher welfare. In this sense nominal debt can be a burden on optimal monetary policy.

When we consider CRRA preferences with the intertemporal elasticity of substitution different from one, it is still true that in a Markov-perfect the equilibrium path of nominal debt converges to a stationary level of debt. However, it is not zero, but negative or positive, depending on the intertemporal elasticity being greater or lower than one. Furthermore, with such general preferences, optimal sequential policy is time inconsistent even when debt is indexed. The interaction of the two sources of dynamic distortions, resulting from the differing elasticities and from nominal debt, can overturn the above efficiency result and it may actually be the case that nominal debt provides higher welfare than indexed debt. In fact, our computations show that, for relatively low values of debt, welfare is higher when debt is nominal. This is one more illustration of the principle that in a second best adding a distortion may actually increase welfare. However, our computations also show that, for large levels of debt, indexed debt dominates in terms of welfare and, therefore, nominal debt is a burden to monetary policy.

The introduction of additional forms of taxation further clarifies the interplay between the various forms of debt and commitment possibilities. Under the natural assumption that fiscal policy choices are predetermined, we show that the optimal policy problem has the same characterization, provided that the revenues levied through seigniorage are enough to allow for an optimal monetary policy with non-negative interest rates.

If there is full commitment to an optimal fiscal policy, the fiscal authorities, anticipating monetary policy distortions, may choose to fully finance government liabilities and – provided the elasticity of substitution is greater or equal to one – the resulting monetary policy be the Friedman rule of zero nominal interest rates. Moreover, this policy result in the equilibrium that obtains in the economy with full commitment with indexed debt, even if debt is nominal and the monetary authority can not commit.

Ours is a normative (second best) analysis that takes into account the commitment problems which are at the root of institutional design in many developed economies –such as, Central Bank independence, constraints on public indebtness, etc. As such, it brings new light on the ways in which the possibility of monetizing nominal debts can affect monetary policy (a central concern in policy design), and on how optimal debt and monetary policies should be designed. We do not claim that our results on optimal-equilibrium debt paths match –or, should match– observed data. Still, it is the case that the prescriptions of our model could be used to provide a more detailed positive analysis of existing monetary policies and some insights on how monetary and debt policies should be redesigned if necessary.

Regarding the interplay between monetary and fiscal policy, we have confined our analysis to the case in which tax rates are predetermined when monetary policy is



sequentially decided and there is full commitment to fiscal policy decisions. These seem reasonable assumptions, given the institutional constraints and lags through which fiscal and monetary policy operate.

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## Appendices

### 7.1 Equilibria with first-best outcomes

In this Appendix we discuss in more detail equilibria with first-best outcomes; in particular, equilibria where the cash-in-advance constraints may not be binding. To understand these cases it is enough to consider optimal policies with full commitment. In these economies, the first-order conditions of the Ramsey problems in definitions 2 and 3 are given by

$$u_c(c_{t+1}) - \alpha = \lambda \left(1 - \frac{u_c(c_{t+1})}{\alpha}[1-\sigma]\right),\ t \geq 0,$$

and, for period zero, when debt is indexed, by

$$u_c(c_0) - \alpha = \lambda$$

while when debt is nominal by

$$u_c(c_0) - \alpha = \lambda (1 + z_0)$$

with $\lambda$ being the Lagrangean multiplier associated with the implementability constraint

$$\sum_{t=0}^{\infty} \beta^t \left(c_{t+1} u_c(c_{t+1}) \frac{\beta}{\alpha} - (c_t + g)\right) = x_0$$

where $x_0 = b_0$ if debt is indexed, and $x_0 = z_0 c_0$ if debt is nominal. Such implementability constraint assumes that the cash-in-advance constraint is binding and at the optimum, where consumption is constant from period $t = 1$ on, it reduces to

$$\beta \left[\frac{u_c(c_1)}{\alpha} - 1\right] c_1 - (1-\beta)c_0 - g = (1-\beta)x_0$$

Clearly, if initial government assets are large enough such that there is no need to raise distortionary taxes, then $\lambda = 0$ and the first-best is achieved; i.e., $c_1 = c_0 = c^*$, where $\frac{u_c(c^*)}{\alpha} = 1$.

In both cases, with nominal and indexed debt, there are first-best solutions where the cash-in-advance constraint holds with equality, even if it is not binding. In the indexed debt economy it corresponds to the minimum level of initial assets that can implement the first-best allocation; which is given by $b_0 = b^* \equiv -c^* - \frac{g}{(1-\beta)}$. When initial assets are larger ($b_0 < b^*$), the government lump sum transfers to the consumers the redundant assets ($b^* - b_0$) in order to implement the first-best allocation. In the nominal debt economy there is also an equilibrium implementing the first-best allocation with the



cash-in-advance constraints holding with equality when $z_0 \equiv z^* = -1 - \frac{g}{(1-\beta)c^*}$; which results in real initial assets $b^* = z^* c^*$.

In the case with nominal debt, it is possible to implement the first-best allocations for any $z_0 < -1$, i.e., $M_0^g + B_0^g(1+i_0) < 0$, although then the cash-in-advance constraints do not hold with equality and therefore the implementability conditions cannot be written as above, replacing the cash-in-advance constraints with equality. By imposing the transversality condition (10), the budget constraint of the government (12 ) can then be written as

$$\sum_{t=0}^{\infty} \beta^{t+1} \frac{i_{t+1} M_{t+1}^g}{P_{t+1}} = \frac{M_0^g + B_0^g(1+i_0)}{P_0} + \frac{g}{1-\beta}$$

But if $M_0^g + B_0^g(1+i_0) < 0$, there is a price $P_0$, such that the first best $c_t = c^*$ with $\frac{u_c(c_t)}{\alpha} = 1$, for all $t \geq 0$, is achieved. At the first best $i_{t+1} = 0$, $t \geq 0$, and therefore

$$\frac{M_0^g}{P_0}(1 + z_0) = -\frac{g}{1-\beta}$$

If $z_0 \in (z^*, -1)$ then $\frac{M_0^g}{P_0} > c^*$ ; i.e., the cash-in-advanced constraint is not satisfied with equality. In particular, if $1 + z_0 = -\varepsilon < 0$, with $\varepsilon \to 0$, in order to achieve the first best, $\frac{M_0^g}{P_0} \to \infty$, which means that initial real debt is $\frac{B_0^g(1+i_0)}{P_0} = -\frac{M_0^g}{P_0} - \frac{g}{1-\beta} \to -\infty$. If $z_0 < z^*$ then the government can implement the first-best allocation corresponding to $z^*$, with equilibrium price $P_0^* = \frac{M_0^g}{c^*}$, by giving lump-sum transfers $P_0^*(z^* - z_0)$.

The solution with total assets that are positive but arbitrarily low is not an equilibrium if anticipated. Indeed from the Fisher equation (8), the nominal interest rate would have to be negative and approaching $-1$, as the price level approaches zero. Private agents would be able to make infinite profits borrowing at the negative nominal rate and holding money. As a result, when $z_0 < -1$, but close to $-1$, there is a first-best *full commitment Ramsey equilibrium,* but there is no *Markov-perfect equilibrium* for $z < -1$ and close to $-1$, since the later imposes that the government policy must be anticipated, and the above incentive to deflate is equally present when there is no commitment.

Finally, the case $z_0 = -1$ also deserves to be discussed. The corresponding *full commitment Ramsey equilibrium* results in a time-inconsistent optimal policy given by $c_0 = c^*$ and $c_1$ being the solution to $\beta \left[\frac{u_c(c_1)}{\alpha} - 1\right] c_1 = g$, while $z_t = -1$, for $t \geq 1$. A *Markov-perfect equilibrium* with $z_0 = -1$ must have $c_0 = c^*$ but $z_1 > -1$, since it is not possible to have $z_1 \leq -1$.

### 7.2 Numerical Exercises

We carry out three numerical exercises for three different values of the elasticity of substitution, corresponding to $\sigma_1 = 0.6$, $\sigma_2 = 1.0$ and $\sigma_3 = 1.4$. To solve our model economies we must choose numerical values for $\alpha$, $\beta$, and $g$. These are the same for the different economies. We assume that the value of $\beta$ is such that the real interest rate is approximately two percent. Consequently, $\beta = 0.98$.

We take as reference values a constant government expenditures to output ratio of $g/y = 0.01$ and a constant debt to output ratio of $b/y = 0.8$. The reason for the low expenditures to output ratio is that those are the expenditures to be financed with seigniorage which is a relatively low share of tax revenues. Since in our model economies $g + c = y$, these choices imply that $g/c = 0.1$ and $b/c = 0.81$. Next, we normalize units so that $c = 1$ and, therefore $g = 0.1$ and $y = 1.01$. To obtain the value of $\alpha$ we use the implementability condition, (14), with stationary values of consumption and debt:

$$\frac{c^{-\sigma}}{\alpha} = \frac{1}{\beta}\left[1 + \frac{g}{c} + (1-\beta)\frac{b}{c}\right] \tag{43}$$

Notice that, given our normalization $c = 1$, different values of $\sigma$ result in the same choice of $\alpha$; which, given the rest of parameters, is $\alpha = 0.95$.

These choices imply that the values of $b$ that support the first best in the economies with indexed debt are $b_1^* = -1.59$, $b_2^* = -1.55$, $b_3^* = -1.54$. These are obtained by computing $c_i^*$ such that $\frac{(c_i^*)^{-\sigma_i}}{\alpha} = 1$, and the corresponding value of $b_i^*$ satisfying (43). The choices of parameters also imply that the steady-state values of $z$ for the economies with nominal debt are $\bar{z}_1 = -0.429$ (corresponding to $\bar{c}_1 = 1.05$ and $\bar{b}_1 = -0.451$), $\bar{z}_2 = 0$ (corresponding to $\bar{c}_2 = 1.02$ and $\bar{b}_2 = 0$), $\bar{z}_3 = 0.419$ (corresponding to $\bar{c}_3 = 1.01$ and $\bar{b}_3 = 0.423$). These are obtained using (20) for stationary consumption and the implementability condition (11) in the steady state, so that

$$\bar{z}_0 = -\frac{u_c(\bar{c})}{\alpha}(1-\sigma)$$

and

$$\frac{\bar{c}u_c(\bar{c})}{\alpha}\left[1 - (1-\beta)\sigma\right] = \bar{c} + g.$$

Therefore, since $(1-\beta)\sigma < 1$, there is a solution for $\bar{c}$ and, correspondingly, for $\bar{z}_0$.

**Algorithm**

Let $u(c) = (c^{1-\sigma} - 1)/(1-\sigma)$, then to compute the monetary equilibria numerically, we solve the following dynamic programs:

$$V(x) = \max\left\{u(c) - \alpha(c+g) + \beta V(x')\right\} \tag{44}$$

subject to

$$\frac{\beta}{\alpha}C(b')^{1-\sigma} + \beta b' = c + g + b \tag{45}$$



when $x = b$ and the debt is indexed, or subject to

$$\frac{\beta}{\alpha}C(z')^{1-\sigma} + \beta z' C(z') = (1+z)c + g \tag{46}$$

when $x = z$ and the debt is nominal

The Bellman operators associated with these problems are:

$$V_{n+1}(x) = T[V_n(x)] = \max\left\{u(c) - \alpha(c+g) + \beta V_n(x')\right\} \tag{47}$$

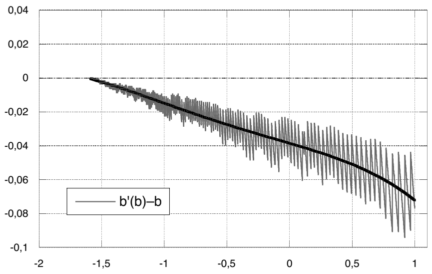
subject to expression (45) when $x = b$ and the debt is indexed, or subject to expression (46) when $x = z$ and the debt is nominal.

To solve these problems, we use the following algorithm:

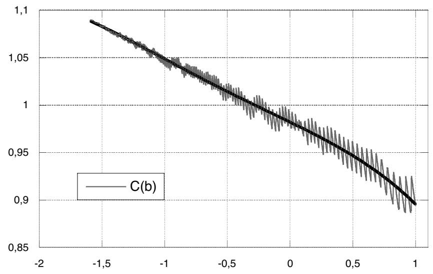
- Step 1: Choose numerical values for parameters $\alpha$, $\beta$, $\sigma$, and $g$.
- Step 2: Define a discrete grid on $x$ (with the first best level of debt as a lower bound for the grid with indexed debt, $-1$ as a lower bound for the grid with nominal debt, as well as for the grid of welfare comparisons, and a large upper bound, as to capture an unambiguous welfare ranking across regimes for high values of debt. This may require robustness test on the upper bound; see below how we treat the case $\sigma > 1$)
- Step 3: Define a decreasing discrete function $C_n(x)$
- Step 4: Define an initial discrete function $V_n(x)$ and iterate on the Bellman operator defined above until we find the converged $V^*(x)$, $x'^*(x)$, $C^*(x)$
- Step 5: If $C^*(x) = C_n(x)$, we are done. Else, update $C_n(x)$ and go to Step 3.

The previous algorithm must be modified to compute the indexed economies with $\sigma > 1$ where the level of debt grows (at the rate lower than $\beta^{-1}$). In this case we iterate the above procedure by expanding the upper bound (in Step 2) until successive iterations do not (significantly) change optimal policies in the relevant range.

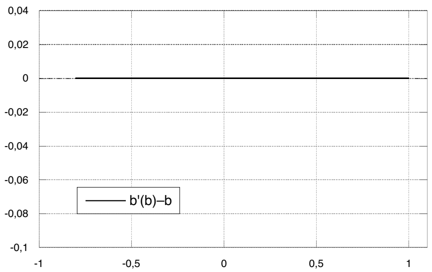
Figure 1: Indexed Debt for various values of $\sigma$



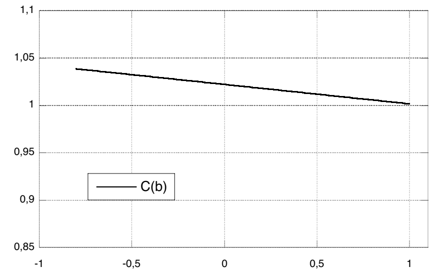

A. Indexed Debt: $b'(b) - b$ for $\sigma = 0.6$



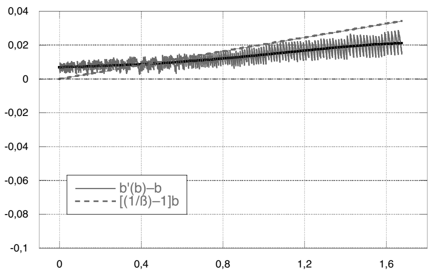

B. Indexed Debt: $C(b)$ for $\sigma = 0.6$



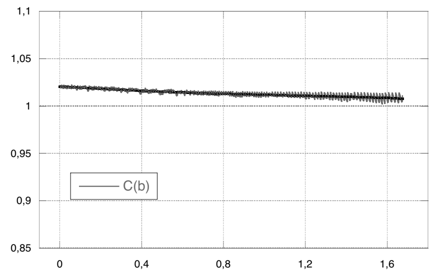

C. Indexed Debt: $b'(b) - b$ for $\sigma = 1.0$




D. Indexed Debt: $C(b)$ for $\sigma = 1.0$



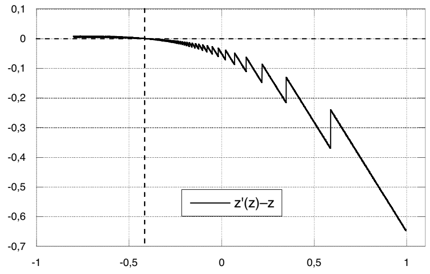

E. Indexed Debt: $b'(b) - b$ for $\sigma = 1.4$



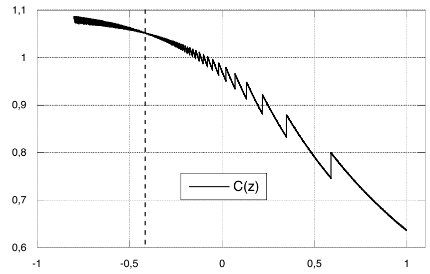

F. Indexed Debt: $C(b)$ for $\sigma = 1.4$



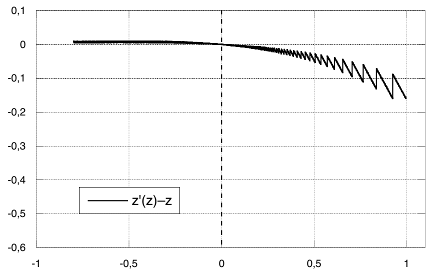
Figure 2: Nominal Debt for various values of $\sigma$



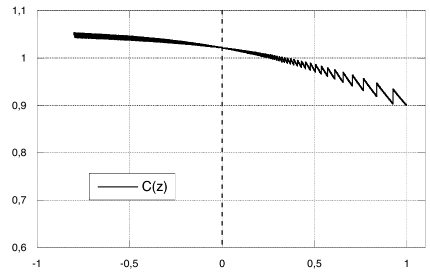

A. Nominal Debt: $z'(z) - z$ for $\sigma = 0.6$



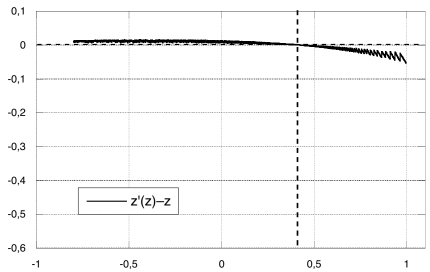

B. Nominal Debt: $C(z)$ for $\sigma = 0.6$



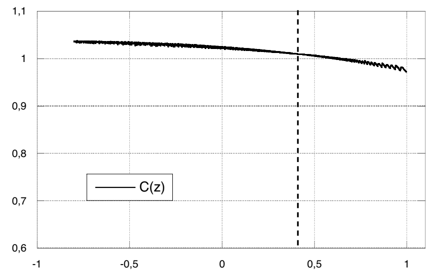

C. Nominal Debt: $z'(z) - z$ for $\sigma = 1.0$




D. Nominal Debt: $C(z)$ for $\sigma = 1.0$

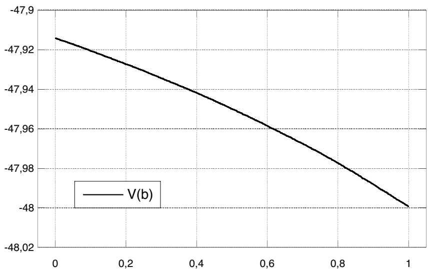



E. Nominal Debt: $z'(z) - z$ for $\sigma = 1.4$

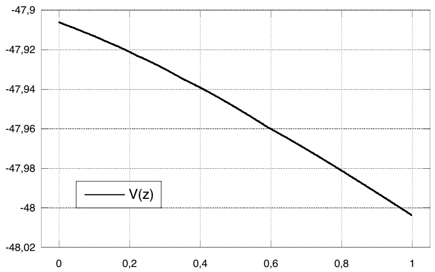



F. Nominal Debt: $C(z)$ for $\sigma = 1.4$

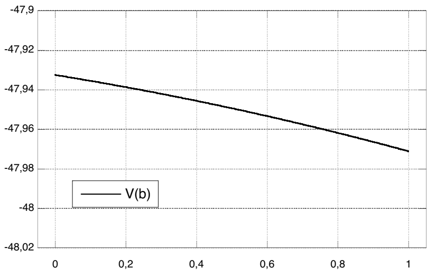
Figure 3: Value Functions for Indexed and Nominal Debt and various values of $\sigma$



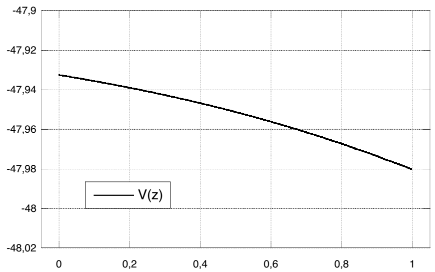

A. Indexed Debt: $V(b)$ for $\sigma = 0.6$



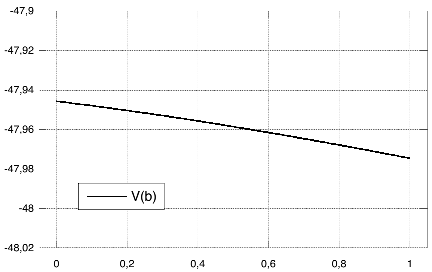

B. Nominal Debt: $V(z)$ for $\sigma = 0.6$



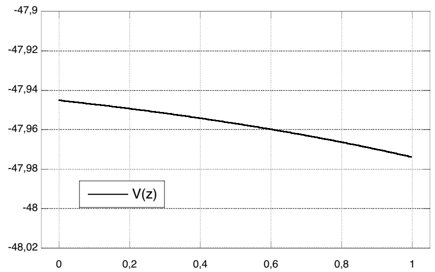

C. Indexed Debt: $V(b)$ for $\sigma = 1.0$




D. Nominal Debt: $V(z)$ for $\sigma = 1.0$



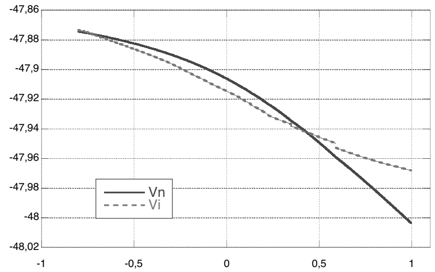

E. Indexed Debt: $V(b)$ $\sigma = 1.4$



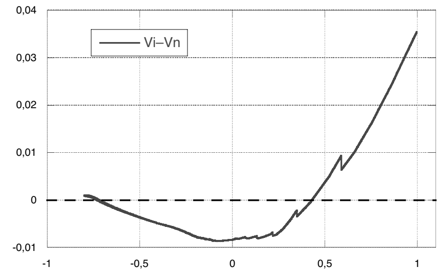

F. Nominal Debt: $V(z)$ for $\sigma = 1.4$



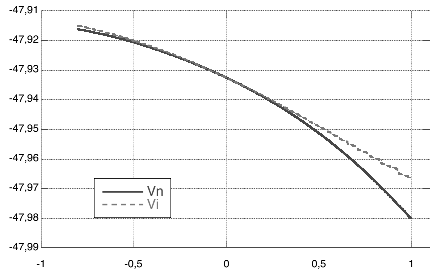
Figure 4: Welfare Comparisons for various values of $\sigma$



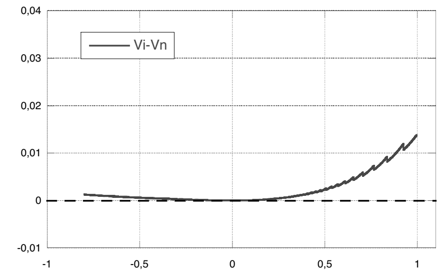

A. $V_n(z)$ and $V_i[b(z)]$ for $\sigma = 0.6$



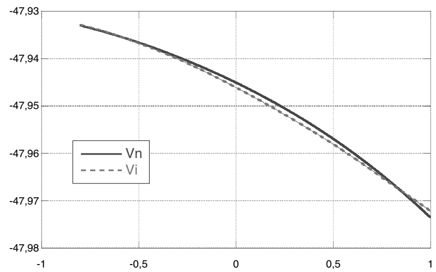

B. $V_i[b(z)] - V_n(z)$ for $\sigma = 0.6$



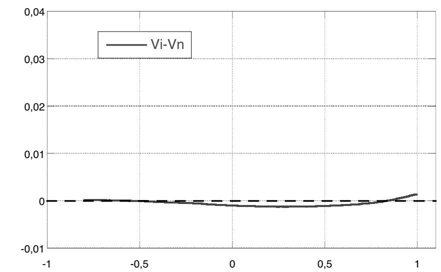

C. $V_n(z)$ and $V_i[b(z)]$ for $\sigma = 1.0$




D. $V_i[b(z)] - V_n(z)$ for $\sigma = 1.0$




E. $V_n(z)$ and $V_i[b(z)]$ for $\sigma = 1.4$




F. $V_i[b(z)] - V_n(z)$ for $\sigma = 1.4$